# Quantum Channel Simulation and the Channel's Smooth Max-Information

(1807.05354)

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CENTRE FOR QUANTUM SOFTWARE AND INFORMATION

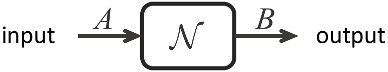
# Quantum channel



In quantum information theory, a quantum channel is a communication channel which can transmit quantum information. It sends one quantum state to the other.

Mathematically, a quantum channel is characterized by a linear map $\mathcal{N}_{A\to B}$ that is

- completely positive **(CP)**: $\mathrm{id}_k \otimes \mathcal{N}(X_{RA}) \geq 0$ for all $X_{RA} \geq 0$ and $k \in \mathbb{N}$;
- trace-preserving **(TP)**: $\operatorname{Tr} \mathcal{N}(X) = \operatorname{Tr} X$ for all $X$.



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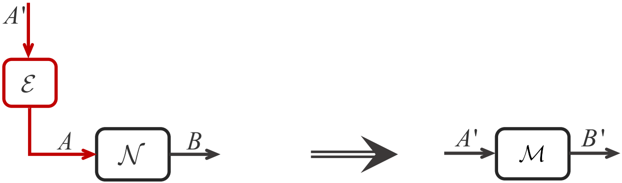

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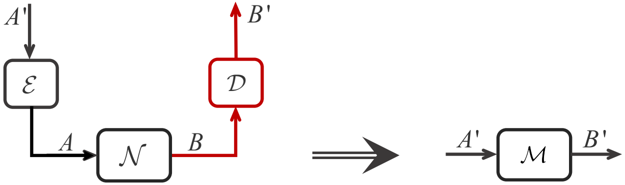

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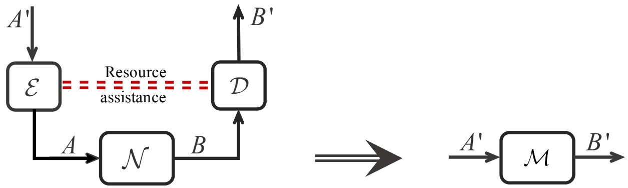

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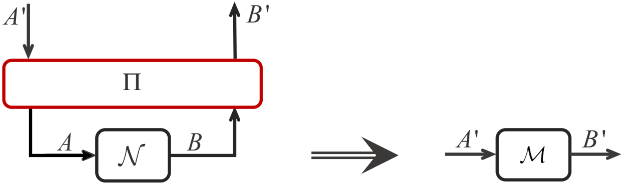

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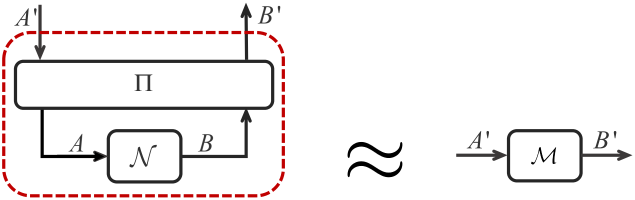

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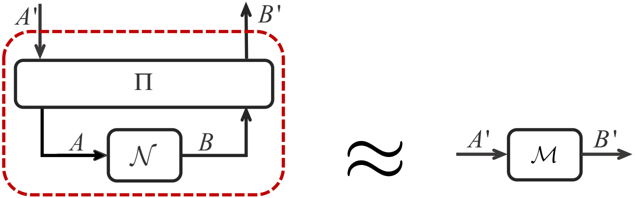
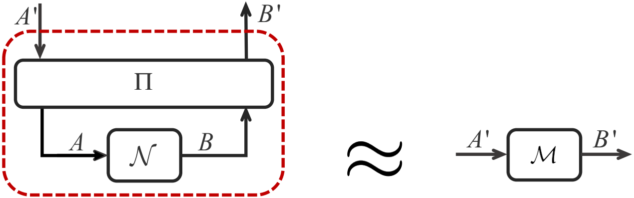



- "Similarity" can be measured via diamond norm [Kitaev, 1997]:

$$\|\mathcal{F}\|_\diamond := \sup_k \|\mathrm{id}_k \otimes \mathcal{F}\|_1, \quad \|\cdot\|_1 \text{induced by the Schatten 1-norm.}$$

- Nice operational meaning: minimum error probability $p_e$ to distinguish two quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ is given by

$$p_e = \frac{1}{2}\left(1 - \frac{\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond}{2}\right).$$




The minimum error of simulation from $\mathcal{N}$ to $\mathcal{M}$ with $\Omega$-assistance is defined as

$$\omega_\Omega(\mathcal{N}, \mathcal{M}) := \frac{1}{2} \inf_{\Pi \in \Omega} \|\Pi \circ \mathcal{N} - \mathcal{M}\|_\diamond . \quad (1)$$

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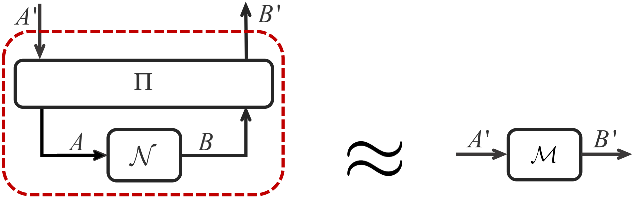


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The channel simulation rate from $\mathcal{N}$ to $\mathcal{M}$ with $\Omega$-assistance is defined as

$$S_\Omega(\mathcal{N}, \mathcal{M}) := \lim_{\varepsilon \to 0} \inf \left\{ \frac{n}{m} : \omega_\Omega(\mathcal{N}^{\otimes n}, \mathcal{M}^{\otimes m}) \leq \varepsilon \right\}. \tag{2}$$

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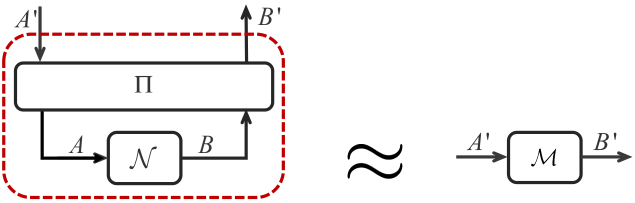


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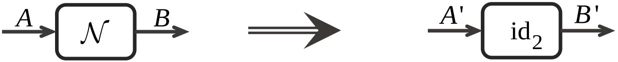
Different resource assistances can be considered. Here we focus on:

- entanglement assistance, $\Omega = \mathrm{E}$;
- no-signalling (NS) assistance, $\Omega = \mathrm{NS}$;
- $\mathrm{E} \subseteq \mathrm{NS}$.

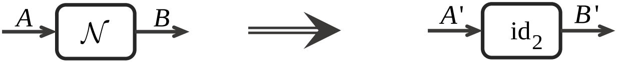



**Q:** What is the optimal rate to simulate the identity channel $\mathrm{id}_2$ via given channel $\mathcal{N}$?

In the framework of channel simulation, we have $Q_\Omega(\mathcal{N}) = S_\Omega(\mathcal{N}, \mathrm{id}_2)^{-1}$.

# Capacity vs. simulation cost



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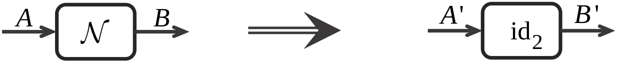


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Operationally, it holds

$$Q_{\mathrm{E}}(\mathcal{N}) \leq Q_{\mathrm{NS}}(\mathcal{N}) \leq S_{\mathrm{NS}}(\mathcal{N}) \leq S_{\mathrm{E}}(\mathcal{N}). \tag{3}$$

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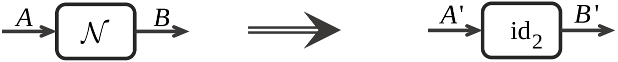


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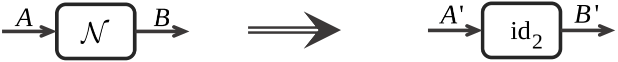


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$$= \frac{1}{2} I(A:B)_{\mathcal{N}} := \frac{1}{2} \max_{\phi_{AA'}} I(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})}$$

$\uparrow$ [Bennett et al., 2002]

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Quantum reverse Shannon theorem (QRTS)

2017 IEEE Information Theory Society Paper Award


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Most the prior results focus on the **asymptotic** simulation rate.

Our contributions are twofolds:

- study the **one-shot** channel simulation task: $\mathrm{id}_m \to \mathcal{N}$ with NS-assistance;
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Recall the minimum error of simulation:

$$\omega_{\mathrm{NS}}(\mathrm{id}_m, \mathcal{N}) := \frac{1}{2} \inf_{\Pi \in \mathrm{NS}} \|\Pi \circ \mathrm{id}_m - \mathcal{N}\|_\diamond . \tag{4}$$

The one-shot quantum simulation cost under NS assistance is defined as

$$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) := \log \min \{ m \in \mathbb{N} : \omega_{\mathrm{NS}}(\mathrm{id}_m, \mathcal{N}) \le \varepsilon \} . \tag{5}$$

Then the asymptotic quantum simulation cost is equivalently given by

$$S_{\mathrm{NS}}(\mathcal{N}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}^{\otimes n}). \tag{6}$$

The minimum error $\omega_{\mathrm{NS}}(\mathrm{id}_m, \mathcal{N})$ can be given by the semidefinite program (SDP),

$$
\begin{aligned}
\text{minimize} \quad & \lambda && \text{(7a)}\\
\text{subject to} \quad & \operatorname{Tr}_{B'} Y_{A'B'} \leq \lambda \mathbb{1}_{A'}, && \text{(7b)}\\
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Sketch of proof: In the definition $\omega_{\text{NS}}(\text{id}_m, \mathcal{N}) := \frac{1}{2} \inf_{\Pi \in \text{NS}} \|\Pi \circ \text{id}_m - \mathcal{N}\|_\diamond$, note that

- $\Pi \in \text{NS}$ if and only if [Leung and Matthews, 2015; Duan and Winter, 2016]
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- $\frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond = \min\{\lambda : \mathrm{Tr}_B Y_{AB} \leq \lambda \mathbb{1}_A, Y_{AB} \geq J_{\mathcal{N}_1} - J_{\mathcal{N}_2}, Y_{AB} \geq 0\}$ [Watrous, 2009].

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- Symmetry of $\mathrm{id}_m$: its Choi matrix is invariant under $U \otimes \overline{U}$ for any unitary $U$.

The one-shot $\varepsilon$-error quantum simulation cost $S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N})$ is given by the following SDP,

$$\frac{1}{2}\log \quad \text{minimize} \quad \operatorname{Tr} V_{B'} \tag{8a}$$

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$$\frac{1}{2}\log \quad \text{minimize} \quad \operatorname{Tr} V_{B'} \tag{8a}$$

$$\text{subject to} \quad \operatorname{Tr}_{B'} Y_{A'B'} \leq \varepsilon \mathbb{1}_{A'}, \tag{8b}$$

$$Y_{A'B'} \geq J_{\widetilde{\mathcal{N}}} - J_{\mathcal{N}},\ Y_{A'B'} \geq 0, \tag{8c}$$

$$J_{\widetilde{\mathcal{N}}} \geq 0,\ \operatorname{Tr}_{B'} J_{\widetilde{\mathcal{N}}} = \mathbb{1}_{A'}, \tag{8d}$$

$$J_{\widetilde{\mathcal{N}}} \leq \mathbb{1}_{A'} \otimes V_{B'}. \tag{8e}$$

---

It reduces to the zero-error case when $\varepsilon = 0$ [Duan and Winter, 2016],

$$S^{(1)}_{\mathrm{NS},0}(\mathcal{N}) = \frac{1}{2}\log\min\{\operatorname{Tr} V_{B'} : J_{\mathcal{N}} \leq \mathbb{1}_{A'} \otimes V_{B'}\} =: \frac{1}{2} H_{\min}(A|B)_{J_{\mathcal{N}}}. \tag{9}$$

What about $\varepsilon > 0$? Do we have

$$\frac{1}{2} I(A:B)_{\mathcal{N}} = S_{\mathrm{NS}}(\mathcal{N}) \xleftarrow{\text{i.i.d.}} S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) \overset{?}{=\!=} \frac{1}{2} H^{\varepsilon}_{\min}(A|B)_{J_{\mathcal{N}}} \xrightarrow{\text{i.i.d.}} \frac{1}{2} H(A|B)_{J_{\mathcal{N}}}$$

We need a one-shot generalization that looks similar to the mutual information.

# Max-information of a quantum state



The quantum mutual information of a state

$$I(A:B)_\rho := \inf_{\sigma_B} D(\rho_{AB} \| \rho_A \otimes \sigma_B). \tag{10}$$

The max-information of a quantum state [Berta et al., 2011]:

$$I_{\max}(A:B)_\rho := \inf_{\sigma_B} D_{\max}(\rho_{AB} \| \rho_A \otimes \sigma_B), \tag{11}$$

where the max-relative entropy [Datta, 2009] $D_{\max}(\rho \| \sigma) := \inf\{t \mid \rho \leq 2^t \cdot \sigma\}$.

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The smoothed version:

$$I^{\varepsilon}_{\max}(A:B)_\rho := \inf_{\widetilde{\rho} \approx^{\varepsilon} \rho} I_{\max}(A:B)_{\widetilde{\rho}}. \tag{12}$$

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$$I_{\max}^{\varepsilon}(A:B)_\rho := \inf_{\widetilde{\rho} \approx^{\varepsilon} \rho} I_{\max}(A:B)_{\widetilde{\rho}}. \tag{12}$$

The asymptotic equipartition property (AEP) holds [Berta et al., 2011]:

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} I_{\max}^{\varepsilon}(A:B)_{\rho^{\otimes n}} = I(A:B)_\rho. \tag{13}$$

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◎ We will generalize these notations and results to a **channel's version** and find their connection with the quantum channel simulation task.

## Definition

For any quantum channel $\mathcal{N}_{A'\to B}$ we define the max-information of the channel $\mathcal{N}$ as

$$I_{\max}(A:B)_{\mathcal{N}} := I_{\max}(A:B)_{\mathcal{N}_{A'\to B}(\Phi_{AA'})}, \tag{14}$$

where $\Phi_{AA'}$ is the maximally entangled state.

**Note**: We can replace $\Phi_{AA'}$ to any pure state $\phi_{AA'}$ with Schmidt rank $|A'|$.

## Definition

The channel's smooth max-information is defined by

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}} := \inf_{\substack{\frac{1}{2}\|\widetilde{\mathcal{N}}-\mathcal{N}\|_{\diamond}\leq\varepsilon \\ \widetilde{\mathcal{N}}\in\mathrm{CPTP}(A':B)}} I_{\max}(A:B)_{\widetilde{\mathcal{N}}}, \tag{15}$$

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The channel's smooth max-information is **monotone** under composition with CPTP maps, i.e., for any CPTP maps $\mathcal{N}_{A'_1\to B_1}$, $\mathcal{F}_{A'_0\to A'_1}$ and $\mathcal{T}_{B_1\to B_0}$,

$$I^{\varepsilon}_{\max}(A_0:B_0)_{\mathcal{T}\circ\mathcal{N}\circ\mathcal{F}} \leq I^{\varepsilon}_{\max}(A_1:B_1)_{\mathcal{N}}. \tag{16}$$

## Theorem

*For any quantum channel $\mathcal{N}_{A'\to B}$ and given error tolerance $\varepsilon \geq 0$, we have*

$$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) = \frac{1}{2} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}}. \tag{17}$$

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The channel's smooth max-information has the asymptotic equipartition property,

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = I(A:B)_{\mathcal{N}}. \tag{18}$$

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Sketch of proof:

$$\begin{aligned}
\frac{1}{2}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} &= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n} S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}^{\otimes n}) \\
&= S_{\mathrm{NS}}(\mathcal{N}) && \text{[by definition]} \\
&= Q_E(\mathcal{N}) && \text{[Bennett et al., 2014]} \\
&= \frac{1}{2} I(A:B)_{\mathcal{N}} && \text{[Bennett et al., 2002]}
\end{aligned}$$

# Main results

## Theorem

*For any quantum channel $\mathcal{N}_{A' \to B}$ and given error tolerance $\varepsilon \geq 0$, we have*

$$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) = \frac{1}{2} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}}. \tag{17}$$

The channel's smooth max-information has the asymptotic equipartition property,

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\frac{1}{2} \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} &= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}^{\otimes n}) \\
&= S_{\mathrm{NS}}(\mathcal{N}) && \text{[by definition]} \\
&= Q_E(\mathcal{N}) && \text{[Bennett et al., 2014]} \\
&= \frac{1}{2} I(A:B)_{\mathcal{N}} && \text{[Bennett et al., 2002]}
\end{aligned}$$

**Note**: on the other hand, if we can proof Eq. (18) directly, it implies that

$$Q_{\mathrm{E}}(\mathcal{N}) = Q_{\mathrm{NS}}(\mathcal{N}) = S_{\mathrm{NS}}(\mathcal{N}) \leq S_{\mathrm{E}}(\mathcal{N})$$

$\longrightarrow$ QRST $\longleftarrow$

# Direct proof of AEP "≥"



$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} \quad \geq \quad I(A:B)_{\mathcal{N}}$$

# Direct proof of AEP "≥"

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} \quad \geq \quad I(A:B)_{\mathcal{N}}$$

---

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_{\diamond}\leq\varepsilon} I_{\max}(A:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^{\otimes n}_{AA'})} \qquad \text{[definition]}$$

$$= nI(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})} \qquad \text{[additivity]}$$

$$= nI(A:B)_{\mathcal{N}} \qquad \text{[optimal } \phi_{AA'}\text{]}$$

# Direct proof of AEP "≥"

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} \quad \geq \quad I(A:B)_{\mathcal{N}}$$

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$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n-\mathcal{N}^{\otimes n}\|_\diamond\leq\varepsilon} I_{\max}(A:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^{\otimes n}_{AA'})} \qquad \text{[definition]}$$

$$\overset{[1]}{\geq} \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n-\mathcal{N}^{\otimes n}\|_\diamond\leq\varepsilon} I(A:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^{\otimes n}_{AA'})} \qquad [D_{\max}\geq D]$$

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[1] N. Datta, "Min-and max-relative entropies and a new entanglement monotone", 2009

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} \quad \geq \quad I(A:B)_{\mathcal{N}}$$

---

$$\begin{aligned}
I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} &= \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n-\mathcal{N}^{\otimes n}\|_{\diamond}\leq\varepsilon} I_{\max}(A:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^{\otimes n}_{AA'})} && \text{[definition]}\\
&\overset{[1]}{\geq} \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n-\mathcal{N}^{\otimes n}\|_{\diamond}\leq\varepsilon} I(A:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^{\otimes n}_{AA'})} && [D_{\max}\geq D]\\
&\overset{[2]}{\gtrsim} I(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\phi^{\otimes n}_{AA'})} && \text{[continuity]}\\
&= nI(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})} && \text{[additivity]}\\
&= nI(A:B)_{\mathcal{N}} && [\text{optimal } \phi_{AA'}]
\end{aligned}$$

[1] N. Datta, "Min-and max-relative entropies and a new entanglement monotone", 2009

[2] R. Alicki, M. Fannes, "Continuity of quantum conditional information", 2004

# Post-selection technique

$$\left\| \widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n} \right\|_\diamond = \sup_{\phi^n_{AA'}} \left\| \widetilde{\mathcal{N}}^n_{A'\to B}(\phi^n_{AA'}) - \mathcal{N}^{\otimes n}_{A'\to B}(\phi^n_{AA'}) \right\|_1 \leq \varepsilon$$

# Post-selection technique



$$\left\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\right\|_\diamond = \sup_{\phi^n_{AA'}} \left\|\widetilde{\mathcal{N}}^n_{A'\to B}(\phi^n_{AA'}) - \mathcal{N}^{\otimes n}_{A'\to B}(\phi^n_{AA'})\right\|_1 \le \varepsilon$$

$\Uparrow$ [Christandl, König, Renner 2009]

$$\left\|\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{RAA'}) - \mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{RAA'})\right\|_1 \le \varepsilon(n+1)^{-(|A'|^2-1)}$$

- $\omega^n_{RAA'}$ is a purification of the **de Finetti state**

$$\omega^n_{AA'} := \int \phi^{\otimes n}_{AA'} d(\phi_{AA'}),$$

  $d(\cdot)$ the measure on normalized pure states induced by Haar measure;
- We can make $|R| \le (n+1)^{|A'|^2-1}$. (see e.g. [Berta, Christandl, Renner 2011])

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We are ready to prove

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} \quad \le \quad I(A:B)_{\mathcal{N}}$$

# Direct proof of AEP "≤"

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \le \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[definition]}$$

# Direct proof of AEP "≤"

$$I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \qquad \text{[definition]}$$

$$\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \qquad \text{[post-selection]}$$

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\begin{aligned}
I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} &= \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \le \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[definition]} \\
&\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \le \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[post-selection]} \\
&= \inf_{\substack{\frac{1}{2}\|\sigma^n_{BAR} - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \le \varepsilon \\ \sigma^n_{AR} = \omega^n_{AR}}} I_{\max}(AR:B)_{\sigma^n_{BAR}} && \text{[partial smooth]}
\end{aligned}
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$$I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \le \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[definition]}$$

$$\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \le \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[post-selection]}$$

$$= \inf_{\substack{\frac{1}{2}\|\sigma^n_{BAR} - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \le \varepsilon \\ \sigma^n_{AR} = \omega^n_{AR}}} I_{\max}(AR:B)_{\sigma^n_{BAR}} \quad \text{[partial smooth]}$$

$$\overset{[1]}{\lesssim} I_{\max}^{\varepsilon}(AR:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{A'AR})} \quad \text{[global smooth]}$$

[1] A. Anshu, M. Berta, R. Jain, and M. Tomamichel, To appear, 2018.

# Direct proof of AEP "$\leq$"

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[definition]}$$

$$\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[post-selection]}$$

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$$\overset{[2]}{\lesssim} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{A'A})} \qquad \omega^n_{AA'} := \int \phi^{\otimes n}_{AA'} d(\phi_{AA'}) \quad \text{[get rid of } R\text{]}$$

[1] A. Anshu, M. Berta, R. Jain, and M. Tomamichel, To appear, 2018.

[2] M. Berta, M. Christandl, and R. Renner, "The quantum reverse Shannon theorem based on one-shot information theory", 2011.

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\begin{aligned}
I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} &= \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[definition]}\\
&\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[post-selection]}\\
&= \inf_{\substack{\frac{1}{2}\|\sigma^n_{BAR} - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon \\ \sigma^n_{AR} = \omega^n_{AR}}} I_{\max}(AR:B)_{\sigma^n_{BAR}} && \text{[partial smooth]}\\
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&\overset{[2]}{\lesssim} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{A'A})} \qquad \omega^n_{AA'} := \int \phi^{\otimes n}_{AA'} d(\phi_{AA'}) && \text{[get rid of } R\text{]}\\
&\overset{[2]}{\lesssim} \max_{\phi_{AA'}} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\phi^{\otimes n}_{AA'})} && \text{[quasi-convexity]}
\end{aligned}
$$

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# Direct proof of AEP "≤"



$$I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} = \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \le \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[definition]}$$

$$\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \le \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} \quad \text{[post-selection]}$$

$$= \inf_{\substack{\frac{1}{2}\|\sigma^n_{BAR} - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \le \varepsilon \\ \sigma^n_{AR} = \omega^n_{AR}}} I_{\max}(AR:B)_{\sigma^n_{BAR}} \quad \text{[partial smooth]}$$

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$$\overset{[2]}{\lesssim} \max_{\phi_{AA'}} I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\phi^{\otimes n}_{AA'})} \quad \text{[quasi-convexity]}$$

$$\overset{[2]}{\lesssim} \max_{\phi_{AA'}} nI(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})} \quad \text{[AEP for states]}$$

[1] A. Anshu, M. Berta, R. Jain, and M. Tomamichel, To appear, 2018.

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\begin{aligned}
I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}} &= \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n - \mathcal{N}^{\otimes n}\|_\diamond \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[definition]} \\
&\lesssim \inf_{\frac{1}{2}\|\widetilde{\mathcal{N}}^n(\omega^n_{A'AR}) - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon} I_{\max}(AR:B)_{\widetilde{\mathcal{N}}^n_{A'\to B}(\omega^n_{A'AR})} && \text{[post-selection]} \\
&= \inf_{\substack{\frac{1}{2}\|\sigma^n_{BAR} - \mathcal{N}^{\otimes n}(\omega^n_{A'AR})\|_1 \leq \varepsilon \\ \sigma^n_{AR} = \omega^n_{AR}}} I_{\max}(AR:B)_{\sigma^n_{BAR}} && \text{[partial smooth]} \\
&\overset{[1]}{\lesssim} I_{\max}^{\varepsilon}(AR:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{A'AR})} && \text{[global smooth]} \\
&\overset{[2]}{\lesssim} I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\omega^n_{A'A})} \qquad \omega^n_{AA'} := \int \phi^{\otimes n}_{AA'} d(\phi_{AA'}) && \text{[get rid of } R\text{]} \\
&\overset{[2]}{\lesssim} \max_{\phi_{AA'}} I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}^{\otimes n}_{A'\to B}(\phi^{\otimes n}_{AA'})} && \text{[quasi-convexity]} \\
&\overset{[2]}{\lesssim} \max_{\phi_{AA'}} nI(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})} && \text{[AEP for states]} \\
&= nI(A:B)_{\mathcal{N}} && \text{[definition]}
\end{aligned}
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# Summary


## Channel simulation task

SDP-computable ↑

**One-shot simulation cost under NS assistance**

$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N})$

↓ i.i.d

**QRST under NS assistance**

$(Q_{\mathrm{E}}(\mathcal{N}) =)$
$Q_{\mathrm{NS}}(\mathcal{N}) = S_{\mathrm{NS}}(\mathcal{N})$

## Channel's max-information

SDP-computable ↑ Monotone ↗

**Channel's smooth max-information**

$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}}$

↓ i.i.d

**AEP**

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = I(A:B)_{\mathcal{N}}$$

Thanks for your attention!

See arXiv:1807.05354

for more details

- Second-order asymptotics for $C_{\mathrm{E}}(\mathcal{N})(= 2 \cdot Q_{\mathrm{E}}(\mathcal{N}))$?

  **Q:** What is the optimal rate $r_{n,\varepsilon}$ to reliably transmit classical information via $n$ uses of the quantum channel with entanglement assistance?

  A second-order lower bound has been established [Datta et al., 2016]:

$$r_{n,\varepsilon} \geq C_{\mathrm{E}}(\mathcal{N}) + \sqrt{\frac{V_{\mathrm{E}}(\mathcal{N})}{n}}\, \Phi^{-1}(\varepsilon) + O(\frac{\log n}{n}). \tag{19}$$

  It was conjectured that

$$r_{n,\varepsilon} = C_{\mathrm{E}}(\mathcal{N}) + \sqrt{\frac{V_{\mathrm{E}}(\mathcal{N})}{n}}\, \Phi^{-1}(\varepsilon) + o(\frac{\log n}{n}). \tag{20}$$

  Obtaining the second-order asymptotics of $I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}}$ may provide a matching upper bound and solve the conjecture.

- Other interesting applications of $I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}}$?