Quantum Channel Simulation and the Channel's Smooth Max-Information

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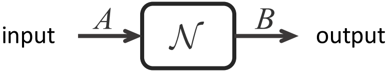
In quantum information theory, a quantum channel is a communication channel which can transmit quantum information. It sends one quantum state to the other.

Mathematically, a quantum channel is characterized by a linear map $\mathcal{N}_{A\to B}$ that is

- completely positive **(CP)**: $\mathrm{id}_k \otimes \mathcal{N}(X_{RA}) \geq 0$ for all $X_{RA} \geq 0$ and $k \in \mathbb{N}$;
- trace-preserving **(TP)**: $\operatorname{Tr} \mathcal{N}(X) = \operatorname{Tr} X$ for all X.



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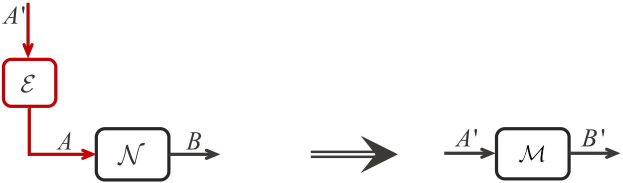

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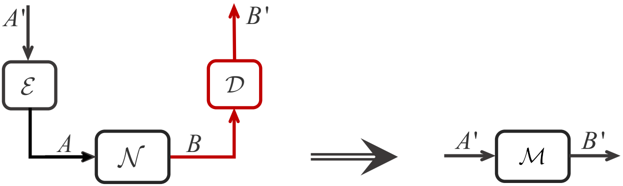

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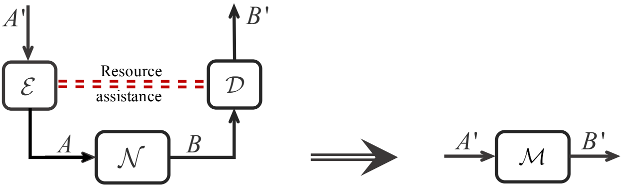

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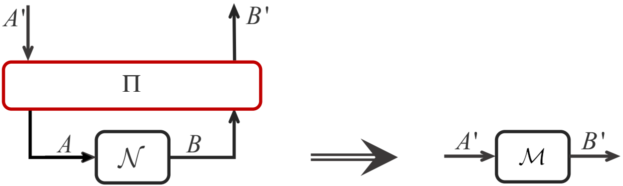

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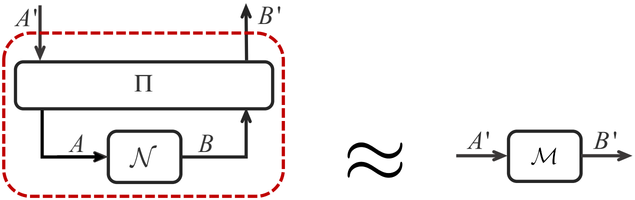

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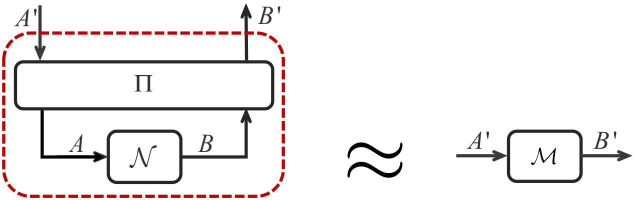

- "Similarity" can be measured via diamond norm [Kitaev, 1997]:

$$\|\mathcal{F}\|_\diamond := \sup_k \|\mathrm{id}_k \otimes \mathcal{F}\|_1, \quad \|\cdot\|_1 \text{induced by the Schatten 1-norm.}$$

- Nice operational meaning: minimum error probability p_e to distinguish two quantum channels $\mathcal{N}_1$ and $\mathcal{N}_2$ is given by

$$p_e = \frac{1}{2}\left(1 - \frac{\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond}{2}\right).$$

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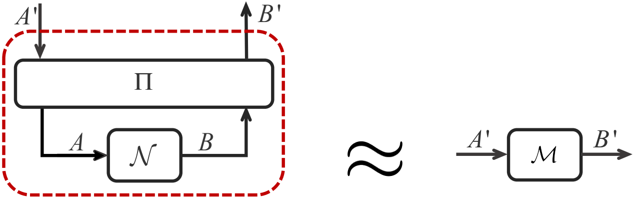

The minimum error of simulation from $\mathcal{N}$ to $\mathcal{M}$ with Ω-assistance is defined as

$$\omega_\Omega(\mathcal{N}, \mathcal{M}) := \frac{1}{2} \inf_{\Pi \in \Omega} \|\Pi \circ \mathcal{N} - \mathcal{M}\|_\diamond. \tag{1}$$

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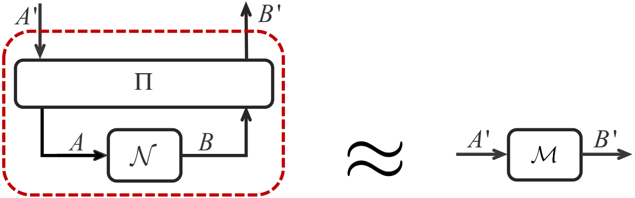

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$$S_\Omega(\mathcal{N},\mathcal{M}) := \lim_{\varepsilon\to 0}\inf\left\{\frac{n}{m} : \omega_\Omega\left(\mathcal{N}^{\otimes n},\mathcal{M}^{\otimes m}\right)\leq\varepsilon\right\}. \tag{2}$$

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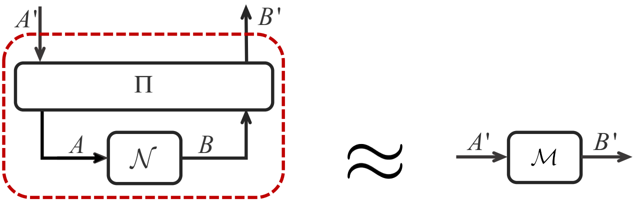

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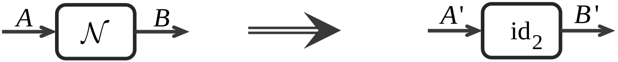
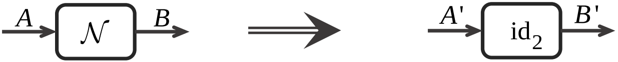
Different resource assistances can be considered. Here we focus on:

- entanglement assistance, $\Omega = \mathrm{E}$;
- no-signalling (NS) assistance, $\Omega = \mathrm{NS}$;
- $\mathrm{E} \subseteq \mathrm{NS}$.



Q: What is the optimal rate to simulate the identity channel id_2 via given channel $\mathcal{N}$?

In the framework of channel simulation, we have $Q_\Omega(\mathcal{N}) = S_\Omega(\mathcal{N}, \mathrm{id}_2)^{-1}$.



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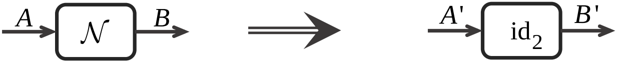

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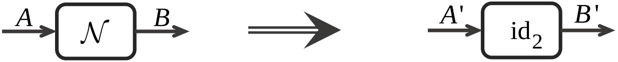
Operationally, it holds

$$Q_E(\mathcal{N}) \leq Q_{NS}(\mathcal{N}) \leq S_{NS}(\mathcal{N}) \leq S_E(\mathcal{N}). \tag{3}$$

Capacity vs. simulation cost



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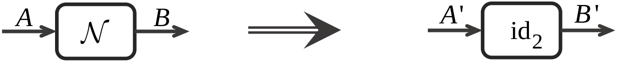

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$$Q_{\mathrm{E}}(\mathcal{N}) \leq Q_{\mathrm{NS}}(\mathcal{N}) \leq S_{\mathrm{NS}}(\mathcal{N}) \leq S_{\mathrm{E}}(\mathcal{N})$$

$$= \frac{1}{2} I(A:B)_{\mathcal{N}} := \frac{1}{2} \max_{\rho_{A'}} I(A:B)_{\mathcal{N}_{A'\to B}(\phi_{AA'})}$$

↑ [Bennett et al., 2002]

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Quantum reverse Shannon theorem (QRTS)

2017 IEEE Information Theory Society Paper Award

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- study the **one-shot** channel simulation scenario, especially $\mathrm{id}_m \to \mathcal{N}$ with NS-assistance;
- introduce a naturally arisen **entropy of a channel**.

Recall the minimum error of simulation:

$$\omega_{\mathrm{NS}}(\mathrm{id}_m, \mathcal{N}) := \frac{1}{2} \inf_{\Pi \in \mathrm{NS}} \|\Pi \circ \mathrm{id}_m - \mathcal{N}\|_\diamond . \tag{4}$$

The one-shot quantum simulation cost under NS assistance is defined as

$$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) := \log \min \{ m \in \mathbb{N} : \omega_{\mathrm{NS}}(\mathrm{id}_m, \mathcal{N}) \le \varepsilon \} . \tag{5}$$

Then the asymptotic quantum simulation cost is equivalently given by

$$S_{\mathrm{NS}}(\mathcal{N}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} S^{(1)}_{\mathrm{NS},\varepsilon}\left(\mathcal{N}^{\otimes n}\right) . \tag{6}$$

The minimum error $\omega_{\mathrm{NS}}(\mathrm{id}_m, \mathcal{N})$ can be given by the semidefinite program (SDP),

$$
\begin{aligned}
\text{minimize} \quad & \lambda && \text{(7a)}\\
\text{subject to} \quad & \mathrm{Tr}_{B'} Y_{A'B'} \leq \lambda \mathbb{1}_{A'}, && \text{(7b)}\\
& Y_{A'B'} \geq J_{\widetilde{\mathcal{N}}} - J_{\mathcal{N}}, \; Y_{A'B'} \geq 0, && \text{(7c)}\\
& J_{\widetilde{\mathcal{N}}} \geq 0, \; \mathrm{Tr}_{B'} J_{\widetilde{\mathcal{N}}} = \mathbb{1}_{A'}, && \text{(7d)}\\
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Sketch of proof: In the definition $\omega_{\rm NS}(\mathrm{id}_m, \mathcal{N}) := \frac{1}{2} \inf_{\Pi \in \rm NS} \|\Pi \circ \mathrm{id}_m - \mathcal{N}\|_\diamond$, note that

- $\Pi \in \rm NS$ if and only if [Leung and Matthews, 2015; Duan and Winter, 2016]
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- $\frac{1}{2}\|\mathcal{N}_1 - \mathcal{N}_2\|_\diamond = \min\left\{\lambda : \operatorname{Tr}_B Y_{AB} \le \lambda \mathbb{1}_A, Y_{AB} \ge J_{\mathcal{N}_1} - J_{\mathcal{N}_2}, Y_{AB} \ge 0\right\}$ [Watrous, 2009].

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- Symmetry of id_m: its Choi matrix is invariant under $U \otimes \overline{U}$ for any unitary U.

The one-shot ε-error quantum simulation cost $S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N})$ is given by the following SDP,

$$\frac{1}{2}\log \quad \text{minimize} \quad \operatorname{Tr} V_{B'} \tag{8a}$$

$$\text{subject to} \quad \operatorname{Tr}_{B'} Y_{A'B'} \leq \varepsilon \mathbb{1}_{A'}, \tag{8b}$$

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It reduces to the zero-error case when $\varepsilon = 0$ [Duan and Winter, 2016],

$$S^{(1)}_{\mathrm{NS},0}(\mathcal{N}) = \frac{1}{2}\log\min\left\{\operatorname{Tr} V_{B'} : J_{\mathcal{N}} \leq \mathbb{1}_{A'} \otimes V_{B'}\right\} =: \frac{1}{2} H_{\min}(A|B)_{J_{\mathcal{N}}}. \tag{9}$$

Since the zero-error cost is additive, we have

$$S_{\mathrm{NS},0}(\mathcal{N}) := \lim_{n\to\infty} \frac{1}{n} S^{(1)}_{\mathrm{NS},0}\left(\mathcal{N}^{\otimes n}\right) = S^{(1)}_{\mathrm{NS},0}(\mathcal{N}). \tag{10}$$

The max-information of a quantum state [Berta et al., 2011]:

$$I_{\max}(A:B)_\rho := \inf_{\sigma_B} D_{\max}\left(\rho_{AB} \| \rho_A \otimes \sigma_B\right), \tag{11}$$

where the max-relative entropy [Datta, 2009] $D_{\max}(\rho\|\sigma) := \log \inf\{t \mid \rho \le t \cdot \sigma\}$.

The smoothed version:

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The asymptotic equipartition property (AEP) holds [Berta et al., 2011]:

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\rho^{\otimes n}} = I(A:B)_\rho, \tag{13}$$

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- $I^{\varepsilon}_{\max}(A:B)_\rho$ is useful in quantum information tasks (state redistribution...)
- We will generalize these notations and results to a **channel's version** and find their connection with the quantum channel simulation task.

Definition

For any quantum channel $\mathcal{N}_{A'\to B}$ we define the max-information of the channel $\mathcal{N}$ as

$$I_{\max}(A:B)_{\mathcal{N}} := I_{\max}(A:B)_{\mathcal{N}_{A'\to B}(\Phi_{AA'})}, \tag{14}$$

where $\Phi_{AA'}$ is the maximally entangled state.

Note: We can replace $\Phi_{AA'}$ to any pure state $\phi_{AA'}$ with Schmidt rank $|A'|$.

Definition

The channel's smooth max-information is defined by

$$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}} := \inf_{\substack{\frac{1}{2}\|\widetilde{\mathcal{N}}-\mathcal{N}\|_{\diamond}\leq\varepsilon \\ \widetilde{\mathcal{N}}\in \text{CPTP}(A':B)}} I_{\max}(A:B)_{\widetilde{\mathcal{N}}}, \tag{15}$$

The channel's smooth max-information is **monotone** under composition with CPTP maps, i.e., for any CPTP maps $\mathcal{N}_{A'_1\to B_1}$, $\mathcal{F}_{A'_0\to A'_1}$ and $\mathcal{T}_{B_1\to B_0}$,

$$I^{\varepsilon}_{\max}(A_0:B_0)_{\mathcal{T}\circ\mathcal{N}\circ\mathcal{F}} \leq I^{\varepsilon}_{\max}(A_1:B_1)_{\mathcal{N}}. \tag{16}$$

Theorem

For any quantum channel $\mathcal{N}_{A' \to B}$ and given error tolerance $\varepsilon \geq 0$, we have

$$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N}) = \frac{1}{2} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}}\,. \tag{17}$$

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The channel's smooth max-information has the asymptotic equipartition property,

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = I(A:B)_{\mathcal{N}} \,, \tag{18}$$

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Sketch of proof:

$$\begin{aligned} \frac{1}{2} \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} &= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} S^{(1)}_{\mathrm{NS},\varepsilon}\left(\mathcal{N}^{\otimes n}\right) \\ &= S_{\mathrm{NS}}(\mathcal{N}) && \text{[by definition]} \\ &= Q_E(\mathcal{N}) && \text{[Bennett et al., 2014]} \\ &= \frac{1}{2} I(A:B)_{\mathcal{N}} && \text{[Bennett et al., 2002]} \end{aligned}$$

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\end{aligned}$$

Note: on the other hand, if we can proof Eq. (18) directly, it implies that

$$Q_{\mathrm{E}}(\mathcal{N}) = Q_{\mathrm{NS}}(\mathcal{N}) = S_{\mathrm{NS}}(\mathcal{N}). \tag{19}$$

Channel simulation task

SDP-computable

One-shot simulation cost under NS assistance

$S^{(1)}_{\mathrm{NS},\varepsilon}(\mathcal{N})$

i.i.d

QRST under NS assistance

$(Q_{\mathrm{E}}(\mathcal{N}) =)$

$Q_{\mathrm{NS}}(\mathcal{N}) = S_{\mathrm{NS}}(\mathcal{N})$

Channel's max-information

SDP-computable

Monotone

Channel's smooth max-information

$I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}}$

i.i.d

AEP

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = I(A:B)_{\mathcal{N}}$$

Discussions

- Directly prove the AEP

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}I^{\varepsilon}_{\max}(A:B)_{\mathcal{N}^{\otimes n}} = I(A:B)_{\mathcal{N}}. \qquad \checkmark \quad \text{online soon}$$

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- Second-order asymptotics for $C_E(\mathcal{N}) (= 2 \cdot Q_E(\mathcal{N}))$?

Q: What is the optimal rate $r_{n,\varepsilon}$ to reliably transmit classical information via n uses of the quantum channel with entanglement assistance?

A second-order lower bound has been established [Datta et al., 2016]:

$$r_{n,\varepsilon} \geq C_E(\mathcal{N}) + \sqrt{\frac{V_E(\mathcal{N})}{n}}\,\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right). \tag{20}$$

It was conjectured that

$$r_{n,\varepsilon} = C_E(\mathcal{N}) + \sqrt{\frac{V_E(\mathcal{N})}{n}}\,\Phi^{-1}(\varepsilon) + o\left(\frac{\log n}{n}\right). \tag{21}$$

Obtaining the second-order asymptotics of $I_{\max}^{\varepsilon}(A:B)_{\mathcal{N}}$ may provide a matching upper bound and solve the conjecture.

- Directly prove the AEP

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} I_{\max}^{\varepsilon} (A : B)_{\mathcal{N}^{\otimes n}} = I (A : B)_{\mathcal{N}} . \qquad \checkmark \quad \text{online soon}$$

- Second-order asymptotics for $C_{\mathrm{E}} (\mathcal{N}) (= 2 \cdot Q_{\mathrm{E}} (\mathcal{N}))$?

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Obtaining the second-order asymptotics of $I_{\max}^{\varepsilon} (A : B)_{\mathcal{N}}$ may provide a matching upper bound and solve the conjecture.

- Other interesting applications of $I_{\max}^{\varepsilon} (A : B)_{\mathcal{N}}$?

Thanks for your attention!