Distillation of quantum coherence in non-asymptotic settings

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B. Regula, K. Fang, X. Wang, G. Adesso,
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Outline

(1) **Resource theory of quantum coherence**

Free states and operations

Coherence distillation

(2) **Main results**: operational capabilities of free operations

Deterministic one-shot distillation

Probabilistic one-shot distillation

Environment-assisted one-shot distillation

Resource theory of quantum coherence

Free states and operations

Resource theory = **free states** $\mathcal{F}$ + **free operations** $\mathcal{O}$

e.g. entanglement = separable states + LOCC

Coherence: superposition in a given basis $\{|i\rangle\}$

Incoherent pure states: basis states

Incoherent mixed states $\mathcal{I}$: diagonal states $\rho = \Delta(\rho)$

Coherence = incoherent states + incoherent operations

incoherent states + strictly incoherent operations

incoherent states + translationally-invariant oper.

incoherent states + physical incoherent operations

incoherent states + dephasing-covariant operations

incoherent states + genuinely incoherent operations

incoherent states + energy-preserving operations

... overview: Streltsov, Plenio, & Adesso, RMP 2017

Choices of free operations

Maximally incoherent operations (MIO)

$\sigma \in \mathcal{I} \implies \Lambda(\sigma) \in \mathcal{I}$

Dephasing-covariant operations (DIO)

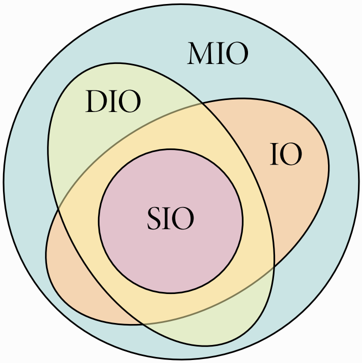
$\Lambda(\Delta(\rho)) = \Delta(\Lambda(\rho)) \ \forall \rho$

Incoherent operations (IO)

$\Lambda(\cdot) = \sum_i K_i \cdot K_i^\dagger, \quad K_i \cdot K_i \in \mathrm{MIO} \ \forall i$

Strictly incoherent operations (SIO)

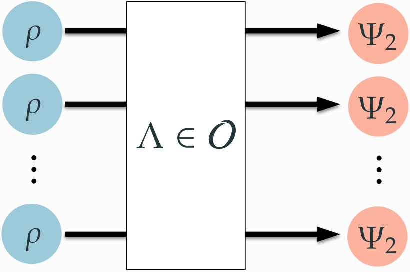
$\Lambda(\cdot) = \sum_i K_i \cdot K_i^\dagger, \quad K_i \cdot K_i \in \mathrm{DIO} \ \forall i$



Maximally coherent state $|\Psi_d\rangle = \sum_{i=0}^{d-1} \frac{1}{\sqrt{d}} |i\rangle$

One-shot coherence distillation

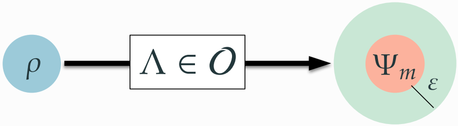
Coherence distillation



Distillable coherence

$$C_{d,\mathcal{O}}^{\infty}(\rho) = \sup_{\Lambda \in \mathcal{O}} \left\{ r \;\middle|\; \lim_{n\to\infty} F(\Lambda\left(\rho^{\otimes n}\right),\ \Psi_2^{\otimes rn}) = 1 \right\}$$

One-shot coherence distillation



One-shot distillable coherence

$$C^{(1),\varepsilon}_{d,\mathcal{O}}(\rho) = \sup_{\Lambda\in\mathcal{O}} \left\{ m \;\middle|\; F\left(\Lambda(\rho),\ \Psi_2^{\otimes m}\right) \geq 1-\varepsilon \right\}$$

$$C^{\infty}_{d,\mathcal{O}}(\rho) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} C^{(1),\varepsilon}_{d,\mathcal{O}}(\rho^{\otimes n})$$

Fidelity of distillation

$$F_{\mathcal{O}}(\rho, m) = \sup_{\Lambda\in\mathcal{O}} F\left(\Lambda(\rho),\ \Psi_m\right)$$

Distillation under MIO and DIO

Theorem

For any state ρ *and operation class* $\mathcal{O} \in \{\mathrm{MIO}, \mathrm{DIO}\}$*, the fidelity of coherence distillation and the one-shot distillable coherence can both be written as the following SDPs:*

$$F_{\mathcal{O}}(\rho, m) = \max\left\{ \operatorname{Tr} G\rho \;\middle|\; 0 \le G \le \mathbb{1},\ \Delta(G) = \frac{1}{m}\mathbb{1} \right\},$$

$$C_{d,\mathcal{O}}^{(1),\varepsilon}(\rho) = -\log \min\left\{ \eta \;\middle|\; \operatorname{Tr} G\rho \ge 1 - \varepsilon, 0 \le G \le \mathbb{1}, \Delta(G) = \eta\mathbb{1} \right\}.$$

Distillation under MIO and DIO

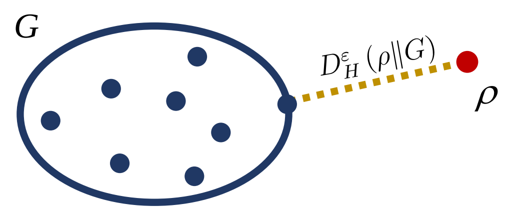
Define $\mathcal{J} = \left\{ G \;\middle|\; \operatorname{Tr} G = 1,\ \Delta(G) = G \right\}$.

Then

$$C_{d,\mathcal{O}}^{(1),\varepsilon}(\rho) = \min_{G \in \mathcal{J}} D_H^{\varepsilon}(\rho \| G) \tag{1}$$

where D_H^{ε} is the hypothesis testing testing relative entropy

$$D_H^{\varepsilon}(\rho\|\sigma) = -\log \min \left\{ \operatorname{Tr} M\rho \;\middle|\; 0 \le M \le \mathbb{1},\ 1 - \operatorname{Tr} M\sigma \le \varepsilon \right\}. \tag{2}$$



Pure-state distillation

$$\left\| |\psi\rangle \right\|_{[m]} = \min_{|\psi\rangle = |x\rangle + |y\rangle} \| |x\rangle \|_{\ell_1} + \sqrt{m} \left\| |y\rangle \right\|_{\ell_2} \tag{3}$$

$$F_{\text{MIO}}(\psi, m) \leq \frac{1}{m} \left\| |\psi\rangle \right\|_{[m]}^2 \qquad \text{by [Regula, 2018]} \tag{4}$$

Since $|\psi\rangle \xrightarrow{SIO} |\psi'\rangle$ iff $|\psi\rangle \prec |\psi'\rangle$ [Chitambar and Gour, 2016],

$$F_{\text{SIO}}(\psi, m) \geq F(\psi, \psi') \quad \text{for any} \quad |\psi\rangle \prec |\psi'\rangle \,. \tag{5}$$

Theorem

For any pure state $|\psi\rangle$, we have

$$F_{\text{SIO}}(\psi, m) = F_{\text{IO}}(\psi, m) = F_{\text{DIO}}(\psi, m) = F_{\text{MIO}}(\psi, m).$$

$$C^{\infty}_{d,\mathcal{O}}(\rho) = \sup_{\Lambda \in \mathcal{O}} \left\{ r \;\middle|\; \lim_{n\to\infty} F\left(\Lambda\left(\rho^{\otimes n}\right), \Psi_2^{\otimes rn}\right) = 1 \right\}$$
$$C^{\infty}_{c,\mathcal{O}}(\rho) = \inf_{\Lambda \in \mathcal{O}} \left\{ r \;\middle|\; \lim_{n\to\infty} F\left(\rho^{\otimes rn}, \Lambda\left(\Psi_2^{\otimes n}\right)\right) = 1 \right\} \tag{6}$$

$$C_r(\rho) := \min_{\sigma \in \mathcal{I}} S(\rho \| \sigma) = S(\rho \| \Delta(\rho))$$

[Winter and Yang, 2016] [Zhao et al., 2018]

$$C_r(\rho)$$

$$C^{\infty}_{d,\mathrm{DIO}}(\rho) \le C^{\infty}_{d,\mathrm{MIO}}(\rho) \le C^{\infty}_{c,\mathrm{MIO}}(\rho) \le C^{\infty}_{c,\mathrm{DIO}}(\rho)$$

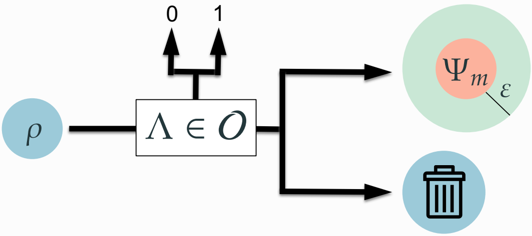
$$C^{\infty}_{d,\mathrm{DIO}}(\rho) = C^{\infty}_{d,\mathrm{MIO}}(\rho) = C^{\infty}_{c,\mathrm{MIO}}(\rho) = C^{\infty}_{c,\mathrm{DIO}}(\rho)$$

$$C_r(\rho)$$

[Zhao et al., 2018; Chitambar, 2018]

Probabilistic coherence distillation

Probabilistic distillation



$$
\begin{aligned}
P_{\mathcal{O}}(\rho \to \Psi_m, \varepsilon) := \text{maximise} \quad & p \\
\text{s.t.} \quad & \Pi_{A \to FB}(\rho) = p\, |0\rangle\langle 0|_F \otimes \rho' + (1-p)\, |1\rangle\langle 1|_F \otimes \omega, \\
& F(\rho', \Psi_m) \geq 1 - \varepsilon, \quad \Pi \in \mathcal{O}.
\end{aligned}
$$

SDP characterisation

Theorem

For any triplet (ρ, m, ε), the maximal success probability of distillation under MIO/DIO are

$$
\begin{aligned}
P_{\mathrm{MIO}}(\rho \to \Psi_m, \varepsilon) = \max.\ & \operatorname{Tr} G\rho \\
\text{s.t.}\ & \Delta(G) = m\Delta(C), \\
& 0 \le C \le G \le \mathbb{1}, \\
& \operatorname{Tr} C\rho \ge (1-\varepsilon) \operatorname{Tr} G\rho.
\end{aligned}
$$

$$
\begin{aligned}
P_{\mathrm{DIO}}(\rho \to \Psi_m, \varepsilon) = \max.\ & \operatorname{Tr} G\rho \\
\text{s.t.}\ & \Delta(G) = m\Delta(C), \\
& 0 \le C \le G \le \mathbb{1}, \\
& \operatorname{Tr} C\rho \ge (1-\varepsilon) \operatorname{Tr} G\rho. \\
& G = \Delta(G).
\end{aligned}
$$

Zero-error case: $P_{\mathrm{MIO}}(\rho \longrightarrow \Psi_m, 0)$

Theorem

For any triplet $(\rho, m, 0)$ with a **full-rank** *state ρ, it holds that $P_{\mathrm{MIO}}(\rho \longrightarrow \Psi_m, 0) = 0$.*

- Any generic density matrix has full rank
- Non-continuity:
  $|P_{\mathrm{MIO}}(\Psi_m^{\varepsilon} \longrightarrow \Psi_m, 0) - P_{\mathrm{MIO}}(\Psi_m \longrightarrow \Psi_m, 0)| = 1$
- Depolarizing noise: $\alpha \cdot \rho + (1 - \alpha)\frac{\mathbb{1}}{m}$ is full rank

Zero-error case: $P_{\mathrm{MIO}}(\rho \to \Psi_m, 0)$

Theorem

For any triplet $(\varphi, m, 0)$ *with a coherent* **pure state** $|\varphi\rangle = \sum_{i=1}^{n} \varphi_i |i\rangle$*, it holds that*

$$P_{\mathrm{MIO}}(\varphi \to \Psi_m, 0) \geq \frac{n^2}{m(\sum_{i=1}^{n} |\varphi_i|^{-2})} > 0.$$

Fundamental difference between MIO and DIO,
in contrast to the deterministic case:

$$P_{\mathrm{MIO}}(\Psi_n \to \Psi_{n+1}, 0) \geq \tfrac{n-1}{n} \to 1;$$
$$P_{\mathrm{DIO}}(\Psi_n \to \Psi_{n+1}, 0) = 0.$$

Zero-error case: $P_{\mathrm{DIO}}(\rho \longrightarrow \Psi_m, 0)$

For any pure state $|\varphi\rangle = \sum_{i=1}^{n} \sqrt{\varphi_i}\,|i\rangle$, it holds that

[Vidal, 1999] and [Chitambar and Gour, 2016]

$$P_{\mathrm{SIO}}(\varphi \longrightarrow \Psi_m, 0) = \begin{cases} 0 & \text{if } \operatorname{rank} \Delta(\varphi) < m, \\ \min_{k\in[1,m]} \frac{m}{k} \sum_{i=m-k+1}^{d} \varphi_i & \text{otherwise.} \end{cases} \quad (9)$$

Theorem

For any pure state φ and any m, we have

$$P_{\mathrm{DIO}}(\varphi \longrightarrow \Psi_m, 0) = P_{\mathrm{SIO}}(\varphi \longrightarrow \Psi_m, 0). \quad (10)$$

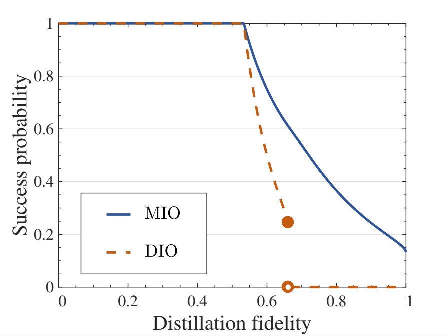
Fidelity and probability of distillation

Is there a natural trade-off between $P_{\mathcal{O}}$ and error ε?

Theorem

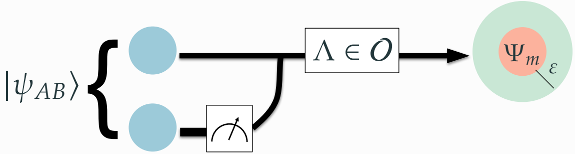
For any pure state $|\varphi\rangle = \sum_{i=1}^{n} \varphi_i |i\rangle$*, it holds that*

$$P_{\mathrm{DIO}}(\varphi \to \Psi_m, \varepsilon) \begin{cases} > 0 & \text{if } n \geq m \text{ or if } n < m \text{ and } \varepsilon \geq 1 - \frac{n}{m}, \\ = 0 & \text{if } n < m \text{ and } \varepsilon < 1 - \frac{n}{m}. \end{cases}$$



Assisted coherence distillation

Assisted distillation



(1) Bob performs POVM

(2) Bob communicates the result to Alice

(3) Alice distills coherence

Optimal measurement can be taken to be pure [Buscemi and Datta, 2013]

Fidelity of assisted distillation

$$C_{a,\mathcal{O}}^{(1),\varepsilon}(\rho) = \sup_{\Lambda \in \mathcal{O}} \left\{ m \;\middle|\; F_{a,\mathcal{O}}(\rho_A, m) \geq 1 - \varepsilon \right\} \tag{11}$$

$$\begin{aligned} F_{a,\mathcal{O}}\left(\rho_A, m\right) &= \max_{\substack{\rho_A = \sum_i p_i \psi_i \\ \{\Lambda_i\} \in \mathcal{O}}} F\left(\sum_i p_i \, \Lambda_i\big(\psi_i\big) \,, \Psi_m \right) \\ &= \max_{\rho_A = \sum_i p_i \psi_i} \sum_i p_i F_{\mathcal{O}}(\psi_i, m) \end{aligned}$$

$\Rightarrow$ the same for any set of operations SIO, IO, DIO, MIO

In $d \in \{2, 3\}$, this reduces to an efficiently computable SDP.

Summary

- SDP characterisations for one-shot distillation rate and maximum success probability under MIO and DIO
- Equivalence in pure-state distillation from pure states
- No-go theorem: no full-rank state can be perfectly distilled, even probabilistically
- Operational differences between MIO and DIO in probabilistic distillation
- Characterisation of one-shot assisted distillation — computable in $d = 2, 3$

Recent progress and open questions

IO shown to be approximately equivalent to MIO and DIO in one-shot distillation [Zhao et al., 2018]

SIO significantly less powerful: almost all states are undistillable under SIO! [Lami et al., in prep.]

Q: what is the smallest physically-motivated class of operations which allows for coherence distillation?

Thank you

arxiv:1711.10512
arxiv:1804.09500
arxiv:1807.04705