Semidefinite Programming Converse Bounds for Quantum Communication

Xin Wang, Kun Fang, and Runyao Duan

Abstract—**We derive several efficiently computable converse bounds for quantum communication over quantum channels in both the one-shot and asymptotic regime. First, we derive one-shot semidefinite programming (SDP) converse bounds on the amount of quantum information that can be transmitted over a single use of a quantum channel, which improve the previous bound from [Tomamichel/Berta/Renes, *Nat. Commun.* 7, 2016]. As applications, we study quantum communication over depolarizing channels and amplitude damping channels with finite resources. Second, we find an SDP-strong converse bound for the quantum capacity of an arbitrary quantum channel, which means the fidelity of any sequence of codes with a rate exceeding this bound will vanish exponentially fast as the number of channel uses increases. Furthermore, we prove that the SDP-strong converse bound improves the *partial transposition bound* introduced by Holevo and Werner. Third, we prove that this SDP strong converse bound is equal to the so-called *max-Rains information*, which is an analog to the Rains information introduced in [Tomamichel/Wilde/Winter, *IEEE Trans. Inf. Theory* 63:715, 2017]. Our SDP strong converse bound is weaker than the Rains information, but it is efficiently computable for general quantum channels.**

Index Terms—**Quantum capacity, quantum channel, semidefinite programming, strong converse, quantum coding.**

I. Introduction

A. Background

The reliable transmission of quantum information via noisy quantum channels is a fundamental problem in quantum information theory. The quantum capacity of a noisy quantum channel is the optimal rate at which it can convey quantum bits (qubits) reliably over asymptotically many uses of the channel. The theorem by Lloyd, Shor, and Devetak (LSD) [2]–[4] and the work in [5]–[7] show that the quantum capacity is equal to the regularized coherent information. In general, the regularization of coherent information is necessary since the coherent information can be superadditive. The quantum capacity is notoriously difficult to evaluate since it is characterized by a multi-letter, regularized expression and it is not even known to be computable [8], [9]. Even for the qubit depolarizing channel, the quantum capacity is still unsolved despite substantial effort in the past two decades (see e.g., [10]–[16]). Our understanding of quantum capacity is quite limited, and we even do not know the exact threshold value of the depolarizing noise where the capacity goes to zero.

The converse part of the LSD theorem states that if the rate exceeds the quantum capacity, then the fidelity of any coding scheme cannot approach one in the limit of many channel uses. A strong converse property leaves no room for the trade-off between rate and error, i.e., the error probability vanishes in the limit of many channel uses whenever the rate exceeds the capacity. For classical channels, Wolfowitz [17] established the strong converse property for the classical capacity. For quantum channels, the strong converse property for the classical capacity was confirmed for several classes of channels [18]–[23].

For quantum communication, the strong converse property was studied in [24] and the strong converse of generalized dephasing channels was established [24]. Given an arbitrary quantum channel, a previously known efficiently computable strong converse bound on the quantum capacity for general channels is the partial transposition bound [25], [26]. Recently, the Rains information [24] was established to be a strong converse bound for quantum communication. There are other known upper bounds for quantum capacity [13]–[15], [27]–[31] and most of them require specific settings to be computable and relatively tight.

Moreover, in a practical setting, the number of quantum channel uses is finite, and one has to make a trade-off between the transmission rate and error tolerance. For both practical and theoretical interest, it is important to optimize the trade-off for the rate and infidelity of quantum communication with finite resources. The study of this finite blocklength setting has recently attracted great interest in classical information theory (e.g., [32], [33]) as well as in quantum information theory (e.g., [34]–[40]).

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B. Summary of Results

In this paper, we focus on quantum communication via noisy quantum channels in both the one-shot and asymptotic settings. We study the quantum capacity assisted with positive partial transpose preserving (PPT) and no-signalling (NS) codes [36]. The PPT codes include all the operations that can be implemented by local operations and classical communication while the NS codes are potentially stronger than entanglement-assisted codes.

In section III, we consider the non-asymptotic quantum capacity. We first introduce the one-shot ε-infidelity quantum capacity with PPT-assisted (and NS-assisted) codes and characterize it as an optimization problem. Based on this optimization, we provide semidefinite programming (SDP) bounds to evaluate the one-shot capacity with a given infidelity tolerance. Compared with the previous efficiently computable converse bound given in [40], we show that our SDP converse bounds are tighter in general and can be strictly tighter for basic channels such as the qubit amplitude damping channel and the qubit depolarizing channel.

In section IV, we investigate quantum communication via quantum channels in the asymptotic setting. We first present an SDP strong converse bound, denoted as Q_Γ, on the quantum capacity for a general quantum channel. This bound has some nice properties, such as additivity with respect to tensor products of quantum channels. In particular, Q_Γ is a channel analog of the SDP entanglement measure introduced in [41], and we show here that it is equal to the so-called max-Rains information. This result implies that Q_Γ is no better, in general, as an upper bound on quantum capacity than the Rains information [24]. However, Q_Γ is efficiently computable for general quantum channels. Finally, we show that Q_Γ improves the partial transposition bound [25].

II. Preliminaries

In the following, we will frequently use symbols such as A (or A') and B (or B') to denote (finite-dimensional) Hilbert spaces associated with Alice and Bob, respectively. We use d_A to denote the dimension of system A. The set of linear operators acting on A is denoted by $\mathcal{L}(A)$. The set of positive operators acting on A is denoted by $\mathcal{P}(A)$. The set of positive operators with unit trace is denoted by $\mathcal{S}(A)$, while the set of positive operators with trace no greater than 1 is denoted by $\mathcal{S}_\leq(A)$. We usually write an operator with a subscript indicating the system that the operator acts on, such as M_{AB}, and write $M_A := \operatorname{Tr}_B M_{AB}$. Note that for a linear operator $X \in \mathcal{L}(A)$, we define $|X| = \sqrt{X^\dagger X}$, where $X^\dagger$ is the adjoint operator of X, and the trace norm of X is given by $\|X\|_1 = \operatorname{Tr}|X|$. A quantum channel $\mathcal{N}_{A'\to B}$ is simply a completely positive (CP) and trace-preserving (TP) linear map from $\mathcal{L}(A')$ to $\mathcal{L}(B)$. The Choi-Jamiołkowski matrix of $\mathcal{N}$ is given by $J_\mathcal{N} = \sum_{ij} |i_A\rangle\langle j_A| \otimes \mathcal{N}(|i_{A'}\rangle\langle j_{A'}|)$, where $\{|i_A\rangle\}$ and $\{|i_{A'}\rangle\}$ are orthonormal bases on isomorphic Hilbert spaces A and A', respectively.

A positive semidefinite (PSD) operator $E \in \mathcal{L}(A \otimes B)$ is said to be a positive partial transpose operator (PPT) if $E^{T_B} \geq 0$, where T_B means the partial transpose with respect to the party B, i.e., $(|ij\rangle\langle kl|)^{T_B} = |il\rangle\langle kj|$. A bipartite operation $\Pi_{A_iB_i\to A_oB_o}$ is PPT if and only if its Choi-Jamiołkowski matrix is PPT [42]. The set of PPT operations include all operations that can be implemented by local operations and classical communication.

Semidefinite programming [43] is a useful tool in the study of quantum information and computation with many applications. In this work, we use the CVX software [44] and QETLAB (A Matlab Toolbox for Quantum Entanglement) [45] to solve SDPs.

III. Converse Bounds for Non-Asymptotic Quantum Communication

A. One-Shot ε-Error Capacity and Finite Resource Trade-Off

In this section, we are interested in quantum communication via noisy channels with finite resources. Suppose Alice shares a maximally entangled state Φ_{A_iR} with a reference system R to which she has no access. The goal is to design a quantum coding protocol such that Alice can transfer her share of this maximally entangled state to Bob with very high fidelity. To this end, Alice first performs an encoding operation $\mathcal{E}_{A_i\to A_o}$ on system A_i and then transmits the prepared state through the channel $\mathcal{N}_{A_o\to B_i}$. The resulting state turns out to be $\mathcal{N}_{A_o\to B_i} \circ \mathcal{E}_{A_i\to A_o}(\Phi_{A_iR})$. After Bob receives the state, he performs a decoding operation $\mathcal{D}_{B_i\to B_o}$ on system B_i, where B_o is some system of the same dimension as A_i. The final resulting state will be $\rho_{final} = \mathcal{D}_{B_i\to B_o} \circ \mathcal{N}_{A_o\to B_i} \circ \mathcal{E}_{A_i\to A_o}(\Phi_{A_iR})$. The target of quantum coding is to optimize the fidelity between ρ_{final} and the maximally entangled state Φ_{A_iR}.

One could further imagine the coding protocol as a general super-operator $\Pi_{A_iB_i\to A_oB_o}$. Chiribella *et al.* [46] showed that a two-input and two-output CPTP map $\Pi_{A_iB_i\to A_oB_o}$ sends any CPTP map $\mathcal{N}_{A_o\to B_i}$ to another CPTP map $\mathcal{M}_{A_i\to B_o}$ if and only if $\Pi_{A_iB_i\to A_oB_o}$ is B to A no-signalling (see also [47]). Such bipartite operation Π is called deterministic super-operator or semi-causal quantum operation. Let $\mathcal{M}_{A_i\to B_o}$ denote the resulting composition channel of a deterministic super-operator $\Pi_{A_iB_i\to A_oB_o}$ and a channel $\mathcal{N}_{A_o\to B_i}$. We write $\mathcal{M} = \Pi \circ \mathcal{N}$ for simplicity. Then there exist CPTP maps $\mathcal{E}_{A_i\to A_oC}$ and $\mathcal{D}_{B_iC\to B_o}$, where C is a quantum register, such that [46]–[48]

$$\mathcal{M}_{A_i\to B_o} = \mathcal{D}_{B_iC\to B_0} \circ \mathcal{N}_{A_o\to B_i} \circ \mathcal{E}_{A_i\to A_oC}. \tag{1}$$

The *no-signalling* (NS) codes [36], [47], [49] correspond to the bipartite quantum operations which are no-signalling from B to A and vice-versa. The PPT codes [36] correspond to the deterministic super-operators which are also PPT. The *non-signalling and PPT-preserving* (NS∩PPT) codes correspond to the quantum no-signalling operations which are also PPT. Moreover, a bipartite quantum operation $\Pi_{A_iB_i\to A_oB_o}$ is called unassisted code (UA) if it can be represented as $\Pi_{A_iB_i\to A_oB_o} = \mathcal{E}_{A_i\to A_o} \circ \mathcal{D}_{B_i\to B_o}$. In the following, Ω denotes specific classes of codes, i.e., $\Omega \in \{\text{UA}, \text{NS} \cap \text{PPT}, \text{PPT}\}$.

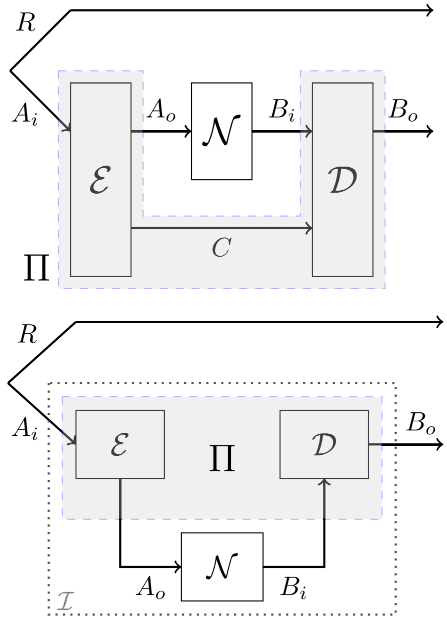

Fig. 1. A deterministic super-operator $\Pi_{A_i B_i \to A_o B_o}$ is equivalently the coding scheme $(\mathcal{E},\mathcal{D})$ with free extra resources such as entanglement. The whole operation aims to simulate a noiseless quantum channel $\mathcal{I}_{A_i \to B_o}$ using a given noisy quantum channel $\mathcal{N}_{A_o \to B_i}$ and the bipartite code Π.

Definition 1: The maximum channel fidelity of $\mathcal{N}$ assisted by the Ω code is defined by

$$F_\Omega(\mathcal{N},k) := \sup_{\Pi} \operatorname{Tr}\left(\Phi_{B_o R} \cdot \Pi_{A_i B_i \to A_o B_o} \circ \mathcal{N}_{A_o \to B_i}\left(\Phi_{A_i R}\right)\right), \tag{2}$$

where $\Phi_{A_i R}$ and $\Phi_{B_o R}$ are maximally entangled states, $k = dim|A_i| = dim|B_o|$ is called code size and the supremum is taken over the Ω codes ($\Omega \in \{\text{UA}, \text{NS} \cap \text{PPT}, \text{PPT}\}$).

Definition 2: For a given quantum channel $\mathcal{N}$ and error tolerance ε, the one-shot ε-error quantum capacity assisted by Ω codes is defined by

$$Q_\Omega^{(1)}(\mathcal{N},\varepsilon) := \log\max\{k \in \mathbb{N} : F_\Omega(\mathcal{N},k) \geq 1-\varepsilon\}, \tag{3}$$

where $\Omega \in \{\text{UA}, \text{NS} \cap \text{PPT}, \text{PPT}\}$. In the following, we write $Q_{\text{UA}}^{(1)}(\mathcal{N},\varepsilon) = Q^{(1)}(\mathcal{N},\varepsilon)$ for simplicity.

The corresponding asymptotic quantum capacity is then given by

$$Q_\Omega(\mathcal{N}) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}Q_\Omega^{(1)}\left(\mathcal{N}^{\otimes n},\varepsilon\right). \tag{4}$$

The authors of [36] showed that the maximum channel fidelity assisted with NS $\cap$ PPT codes is given by the following SDP:

$$\begin{aligned} F_{\text{NS}\cap\text{PPT}}(\mathcal{N},k) = \max\ & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \\ \text{s.t. } & 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}\rho_A = 1, \\ & -k^{-1}\rho_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B}, \\ & W_{AB}^{T_B} \leq k^{-1}\rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}_A W_{AB} = k^{-2}\mathbb{1}_B \ \ (\text{NS}). \end{aligned} \tag{5}$$

To obtain $F_{\text{PPT}}(\mathcal{N},k)$, one only needs to remove the NS constraint.

Combining Eqs. (3) and (5), one can derive the following proposition. It is worth noting that Eq. (6) is not an SDP in general, due to the non-linear term $m\rho_A$ and the condition $\operatorname{Tr}_A W_{AB} = m^2\mathbb{1}_B$. But in next subsection, we will derive several semidefinite relaxations of this optimization problem.

Proposition 3: For any quantum channel $\mathcal{N}_{A' \to B}$ with Choi-Jamiołkowski matrix $J_{\mathcal{N}} \in \mathcal{L}(A \otimes B)$ and given error tolerance ε, its one-shot ε-error quantum capacity assisted with PPT codes can be simplified as the following optimization problem:

$$\begin{aligned} Q_{\text{PPT}}^{(1)}(\mathcal{N},\varepsilon) = -\log\min\ & m \\ \text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, \\ & 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}\rho_A = 1, \\ & -m\rho_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B}, \\ & W_{AB}^{T_B} \leq m\rho_A \otimes \mathbb{1}_B. \end{aligned} \tag{6}$$

If the codes are also non-signalling, we can have the same optimization for $Q_{NS\cap PPT}^{(1)}(\mathcal{N},\varepsilon)$ with the additional constraint $\operatorname{Tr}_A W_{AB} = m^2\mathbb{1}_B$.

B. Improved SDP Converse Bounds for Quantum Communication

To better evaluate the quantum communication rate with finite resources, we introduce several SDP converse bounds for quantum communication with the assistance of PPT (and NS) codes. In Theorem 4, we further prove that our SDP bounds are tighter than the one introduced in [40].

Specifically, Tomamichel *et al.* [40] established that $-\log f(\mathcal{N},\varepsilon)$ is a converse bound on one-shot ε-error quantum capacity, i.e., $Q^{(1)}(\mathcal{N},\varepsilon) \leq -\log f(\mathcal{N},\varepsilon)$ where

$$\begin{aligned} f(\mathcal{N},\varepsilon) = \min\ & \operatorname{Tr} S_A \\ \text{s.t. } & \operatorname{Tr} W_{AB} J_{\mathcal{N}} \geq 1-\varepsilon, \\ & S_A, \Theta_{AB} \geq 0, \operatorname{Tr}\rho_A = 1, \\ & 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\ & S_A \otimes \mathbb{1}_B \geq W_{AB} + \Theta_{AB}^{T_B}. \end{aligned} \tag{7}$$

Here, we introduce a hierarchy of SDP converse bounds on the one-shot ε-error capacity based on the optimization problem in Eq. (6). If we relax the term $m\rho_A$ to a single variable S_A, we will obtain $g(\mathcal{N},\varepsilon)$, where

$$\begin{aligned} g(\mathcal{N},\varepsilon) := \min\ & \operatorname{Tr} S_A \\ \text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, \\ & 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}\rho_A = 1, \\ & -S_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B} \leq S_A \otimes \mathbb{1}_B. \end{aligned} \tag{8}$$

In particular, for the NS condition $\operatorname{Tr}_A W_{AB} = m^2\mathbb{1}_B$, there are two different ways to get relaxations. The first one is to substitute it with $\operatorname{Tr}_A W_{AB} = t\mathbb{1}_B$ and obtain SDP $\widetilde{g}(\mathcal{N},\varepsilon)$).

The second one is to introduce a prior constant $\widehat{m}$ satisfying the inequality

$$Q^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\varepsilon) \le -\log \widehat{m} \tag{9}$$

and then obtain SDP $\widehat{g}(\mathcal{N},\varepsilon)$. Note that the second method can provide a tighter bound, but it requires one more step of calculation since we need to get the prior constant $\widehat{m}$. Successively refining the value of $\widehat{m}$ will result in a tighter bound.

$$\begin{aligned} \widetilde{g}(\mathcal{N},\varepsilon) := \min\ & \operatorname{Tr} S_A \\ \text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \ge 1-\varepsilon, \\ & 0 \le W_{AB} \le \rho_A \otimes \mathbb{1}_B, \operatorname{Tr}\rho_A = 1, \\ & -S_A\otimes \mathbb{1}_B \le W_{AB}^{T_B} \le S_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}_A W_{AB} = t\mathbb{1}_B. \end{aligned} \tag{10}$$

$$\begin{aligned} \widehat{g}(\mathcal{N},\varepsilon) := \min\ & \operatorname{Tr} S_A \\ \text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \ge 1-\varepsilon, \\ & 0 \le W_{AB} \le \rho_A \otimes \mathbb{1}_B, \operatorname{Tr}\rho_A = 1, \\ & -S_A\otimes \mathbb{1}_B \le W_{AB}^{T_B} \le S_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}_A W_{AB} = t\mathbb{1}_B, t \ge \widehat{m}^2. \end{aligned} \tag{11}$$

Theorem 4: For any quantum channel $\mathcal{N}$ and error tolerance ε, the inequality chain holds

$$\begin{aligned} Q^{(1)}(\mathcal{N},\varepsilon) &\le Q^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\varepsilon) && (12) \\ &\le -\log \widehat{g}(\mathcal{N},\varepsilon) \le -\log \widetilde{g}(\mathcal{N},\varepsilon) && (13) \\ &\le -\log g(\mathcal{N},\varepsilon) \le -\log f(\mathcal{N},\varepsilon). && (14) \end{aligned}$$

Proof: The first inequality is trivial. The third and fourth inequalities are easy to obtain since the minimization over a smaller feasible set gives a larger optimal value here.

For the second inequality, suppose the optimal solution of (6) for $Q^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\varepsilon)$, is taken at $\{W_{AB},\rho_A,m\}$. Let $S_A = m\rho_A,\ t = m^2$. Then we can verify that $\{W_{AB},\rho_A,S_A,t\}$ is a feasible solution to the SDP (11) of $\widehat{g}(\mathcal{N},\varepsilon)$. So $\widehat{g}(\mathcal{N},\varepsilon) \le \operatorname{Tr} S_A = m$, which implies that $Q^{(1)}_{\text{NS}\cap\text{PPT}}(\mathcal{N},\varepsilon) = -\log m \le -\log \widehat{g}(\mathcal{N},\varepsilon)$.

For the last inequality, we only need to show that $f(\mathcal{N},\varepsilon) \le g(\mathcal{N},\varepsilon)$. Suppose the optimal solution of $g(\mathcal{N},\varepsilon)$ is taken at $\{\rho_A,S_A,W_{AB}\}$. Let us choose $\Theta_{AB} = S_A\otimes\mathbb{1}_B - W_{AB}^{T_B}$. Since $S_A\otimes\mathbb{1}_B \ge W_{AB}^{T_B}$, it is clear that $\Theta_{AB}\ge 0$ and $S_A\otimes\mathbb{1}_B = W_{AB} + \Theta_{AB}^{T_B}$. Thus, $\{S_A,\rho_A,W_{AB},\Theta_{AB}\}$ is a feasible solution to the SDP (7) of $f(\mathcal{N},\varepsilon)$ which implies $f(\mathcal{N},\varepsilon) \le \operatorname{Tr} S_A = g(\mathcal{N},\varepsilon)$. ∎

C. Examples: Amplitude Damping Channel and Depolarizing Channel

In this subsection, we focus on quantum coding with amplitude damping channels and depolarizing channels. In Fig. 2, we show that for the amplitude damping channel $\mathcal{N}_{AD}$, our converse bound $-\log\widetilde{g}(\mathcal{N},\varepsilon)$ and $-\log g(\mathcal{N},\varepsilon)$ are both tighter than $-\log f(\mathcal{N},\varepsilon)$. For the depolarizing channel $\mathcal{N}_D$, exploiting its symmetry, we further simplify our SDP converse bounds to linear programs.

Example 1: For the amplitude damping channel $\mathcal{N}_{AD} = \sum_{i=0}^{1} E_i\cdot E_i^\dagger$ with $E_0 = |0\rangle\langle 0| + \sqrt{1-r}|1\rangle\langle 1|$, $E_1 = \sqrt{r}|0\rangle\langle 1|$

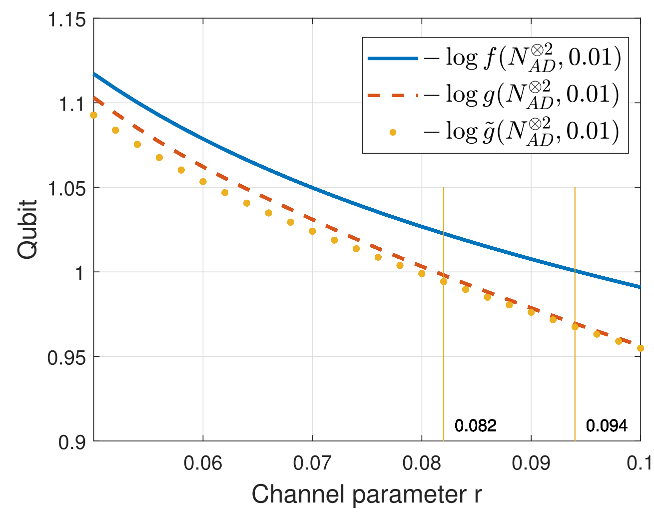

Fig. 2. This figure demonstrates the differences among the SDP converse bounds (i) $-\log f\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right)$ (blue solid), (ii) $-\log g\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right)$ (red dashed), (iii) $-\log\widetilde{g}\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right)$ (yellow dotted), where the channel parameter r ranges from 0.05 to 0.1.

$(0 \le r \le 1)$, the differences among $-\log f\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right)$, $-\log g\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right)$ and $-\log\widetilde{g}\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right)$, are presented in Fig. 2. When $r\in(0.082, 0.094)$, $-\log\widetilde{g}\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right) \le -\log g\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right) < 1 < -\log f\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right)$. It shows that we cannot transmit a single qubit within error tolerance $\varepsilon = 0.01$ via 2 copies of amplitude damping channel where parameter $r\in(0.082, 0.094)$. However, this result cannot be obtained via the converse bound $-\log f\left(\mathcal{N}_{AD}^{\otimes 2}, 0.01\right)$.

Example 2: For the qubit depolarizing channel $\mathcal{N}_D(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$, where X, Y, Z are Pauli matrices, the Choi matrix of $\mathcal{N}_D$ is $J_{\mathcal{N}} = d\left((1-p)\Phi + \frac{p}{d^2-1}\Phi^\perp\right)$, where $d = 2$, $\Phi = \frac{1}{d}\sum_{i,j=0}^{d-1} |ii\rangle\langle jj|$ and $\Phi^\perp = \mathbb{1}_{AB} - \Phi$. For the n-fold tensor product depolarizing channel, its Choi matrix is $J_{\mathcal{N}}^{\otimes n} = d^n\sum_{i=0}^{n} f_i P_i^n\left(\Phi,\Phi^\perp\right)$, where $f_i = (1-p)^i\left(\frac{p}{d^2-1}\right)^{n-i}$ and $P_i^n\left(\Phi,\Phi^\perp\right)$ represent the sum of those n-fold tensor product terms with exactly i copies of Φ. For example,

$$\begin{aligned} P_1^3\left(\Phi,\Phi^\perp\right) &= \Phi^\perp\otimes\Phi^\perp\otimes\Phi + \Phi^\perp\otimes\Phi\otimes\Phi^\perp \\ &\quad + \Phi\otimes\Phi^\perp\otimes\Phi^\perp. \end{aligned} \tag{15}$$

Suppose $\{W_{AB},\rho_A,S_A\}$ is the optimal solution to the SDP (8) for the channel $\mathcal{N}_D^{\otimes n}$, then for any local unitary $U = \bigotimes_{i=1}^n U_A^i \otimes \overline{U}_B^i$, $U_A = \bigotimes_{i=1}^n U_A^i$, we know that $\{UWU^\dagger, U_A\rho_A U_A^\dagger, U_A S_A U_A^\dagger\}$ is also optimal. Convex combinations of optimal solutions remain optimal. Without loss of generality, we can take the optimal solution to be invariant under any local unitary U and U_A, respectively. Again, since $J_{\mathcal{N}}^{\otimes n}$ is invariant under the symmetric group, acting by permuting the tensor factors, we can finally take the optimal solution as $W = \sum_{i=0}^n w_i P_i^n\left(\Phi,\Phi^\perp\right)$, $\rho_A = \mathbb{1}_A/d^n$, $S_A = s\mathbb{1}_A$.

Note that $P_i^n\left(\Phi, \Phi^{\perp}\right)$ are orthogonal projections. Thus without considering degeneracy, operator W has eigenvalues $\{w_i\}_{i=0}^n$. Next, we need to know the eigenvalues of W^{T_B}. Decomposing operators Φ^{T_B} and $\Phi^{\perp T_B}$ into orthogonal projections, i.e.,

$$\Phi^{T_B} = \frac{1}{d}\left(P_+ - P_-\right), \tag{16}$$

$$\Phi^{\perp T_B} = \left(1 - \frac{1}{d}\right) P_+ + \left(1 + \frac{1}{d}\right) P_- \tag{17}$$

where P_+ and P_- are symmetric and anti-symmetric projections respectively and collecting the terms with respect to $P_k^n\left(P_+, P_-\right)$, we have

$$\begin{aligned} W^{T_B} &= \sum_{i=0}^{n} w_i P_i^n\left(\Phi^{T_B}, \Phi^{\perp T_B}\right) && (18)\\ &= \sum_{k=0}^{n}\left(\sum_{i=0}^{n} x_{i,k} w_i\right) P_k^n\left(P_+, P_-\right). && (19)\end{aligned}$$

In the above equation (19), we note that $x_{i,k} = \frac{1}{d^n}\sum_{m=\max\{0,i+k-n\}}^{\min\{i,k\}}\binom{k}{m}\binom{n-k}{i-m}(-1)^{i-m}(d-1)^{k-m}(d+1)^t$ and $t = n - k + m - i$.

Since $P_k^n\left(P_+, P_-\right)$ are also orthogonal projections, W^{T_B} has eigenvalues $\{t_k\}_{k=0}^n$ (without considering degeneracy), where $t_k = \sum_{i=0}^n x_{i,k} w_i$. As for the constraint $\operatorname{Tr} J_{\mathcal{N}}^{\otimes n} W_{AB} \geq 1 - \varepsilon$, we have

$$\begin{aligned} \operatorname{Tr} J_{\mathcal{N}}^{\otimes n} W &= d^n \operatorname{Tr}\sum_{i=0}^{n} f_i w_i P_i^n\left(\Phi, \Phi^{\perp}\right) && (20)\\ &= d^n \sum_{i=0}^{n}\binom{n}{i}(1-p)^i p^{n-i} w_i \geq 1 - \varepsilon. && (21)\end{aligned}$$

Finally, substitute $\eta = s d^n$ and $m_i = w_i d^n$. We obtain the linear program

$$\begin{aligned} g\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right) = \min\ & \eta \\ \text{s.t. } & \sum_{i=0}^{n}\binom{n}{i}(1-p)^i p^{n-i} m_i \geq 1 - \varepsilon, \\ & 0 \leq m_i \leq 1, \quad i = 0, 1, \cdots, n, \\ & -\eta \leq \sum_{i=0}^{n} x_{i,k} m_i \leq \eta, \quad k = 0, 1, \cdots, n. \end{aligned} \tag{22}$$

Following a similar procedure, we have

$$\begin{aligned} f\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right) = \min\ & \eta \\ \text{s.t. } & \sum_{i=0}^{n}\binom{n}{i}(1-p)^i p^{n-i} m_i \geq 1 - \varepsilon, \\ & m_i + s_i \leq \eta, \quad i = 0, 1, \cdots, n, \\ & \eta \geq 0, \quad 0 \leq m_i \leq 1, \ i = 0, 1, \cdots, n \\ & \sum_{i=0}^{n} x_{i,k} s_i \geq 0, \quad k = 0, 1, \cdots, n. \end{aligned}$$

$$\begin{aligned} \widehat{g}\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right) = \min\ & \eta \\ \text{s.t. } & \sum_{i=0}^{n}\binom{n}{i}(1-p)^i p^{n-i} m_i \geq 1 - \varepsilon, \\ & 0 \leq m_i \leq 1, \quad i = 0, 1, \cdots, n, \\ & -\eta \leq \sum_{i=0}^{n} x_{i,k} m_i \leq \eta, \quad k = 0, 1, \cdots, n, \\ & \frac{1}{d^{2n}}\sum_{i=0}^{n}\binom{n}{i}\left(d^2-1\right)^{n-i} m_i \geq \widehat{m}^2. \end{aligned}$$

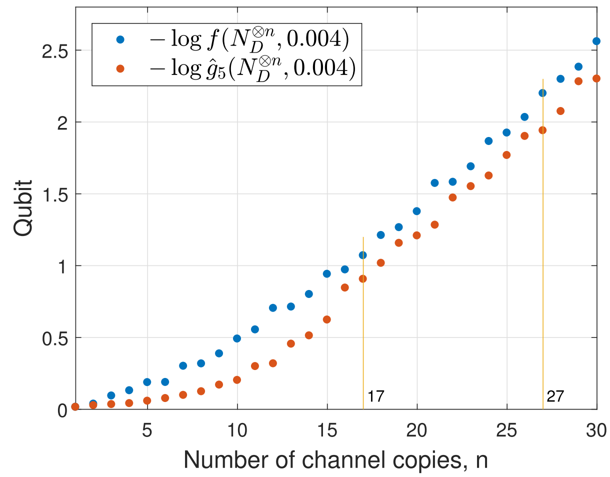

Fig. 3. This figure demonstrates the differences between the SDP converse bounds $-\log f\left(\mathcal{N}_D^{\otimes n}, 0.004\right)$ (blue dots) and $-\log \widehat{g}_5\left(\mathcal{N}_D^{\otimes n}, 0.004\right)$ (red dots), where the channel parameter $p = 0.2$ and the number of channel uses ranges from 1 to 30.

Since $-\log \widehat{g}\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$ is a converse bound for any $\widehat{m} \leq 2^{-Q^{(1)}_{\mathrm{PPT}\cap\mathrm{NS}}(\mathcal{N}_D^{\otimes n}, \varepsilon)}$, we can successively refine the value of $\widehat{m}$ and obtain a tighter result. Let us denote $\widehat{m}_i$ and $\widehat{g}_i\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$ as the values of $\widehat{m}$ and $\widehat{g}\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$ in the i-th iteration, respectively. First, we take the initial value of $\widehat{m}_1 = g\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$ and get the result $\widehat{g}_1\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$. Then we can set $\widehat{m}_{i+1} = \widehat{g}_i\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$ and get the result $\widehat{g}_{i+1}\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$. In Fig. 3, we show that after five iterations, we can get a converse bound $-\log \widehat{g}_5\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$ which is strictly tighter than $-\log f\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$. Especially, when $n = 17$, $-\log \widehat{g}_5\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right) < 1 < -\log f\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$. It shows that we cannot transmit a single qubit within error tolerance $\varepsilon = 0.004$ via 17 copies of depolarizing channel where parameter $p = 0.2$. However, this result cannot be obtained via the converse bound $-\log f\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right)$.

IV. Strong Converse Bound for Quantum Communication

In this section, we establish an SDP strong converse bound Q_Γ (or $R_{\max}$) to evaluate the quantum capacity of a general quantum channel. We summarize our strong converse bound with other well-known bounds in Table I.

A. An SDP Strong Converse Bound on Quantum Capacity

Proposition 5: For any quantum channel $\mathcal{N}$ and error tolerance ε,

$$Q^{(1)}_{\mathrm{PPT}}\left(\mathcal{N}, \varepsilon\right) \leq Q_\Gamma\left(\mathcal{N}\right) - \log\left(1 - \varepsilon\right), \tag{23}$$

TABLE I
COMPARISON OF CONVERSE BOUNDS ON QUANTUM CAPACITY.

	Strong converse rate	Efficiently computable
$Q_\Gamma(R_{\max})$	✓	✓
R	✓	✗ (max-min)
E_C	✓	✗ (regularization)
Q_Θ	✓	✓
Q_E	✓	✓
ε-DEG	?	✓
Q_{ss}	?	? (Unbounded dimension)

where $Q_\Gamma(\mathcal{N}) := \log \Gamma(\mathcal{N})$ and

$$\begin{aligned}
\text{(Primal)}\quad \Gamma(\mathcal{N}) = \max\ & \operatorname{Tr} J_{\mathcal{N}} R_{AB} \\
\text{s.t. } & R_{AB}, \rho_A \geq 0, \operatorname{Tr}\rho_A = 1, \\
& -\rho_A \otimes \mathbb{1}_B \leq R_{AB}^{T_B} \\
& R_{AB}^{T_B} \leq \rho_A \otimes \mathbb{1}_B,
\end{aligned} \tag{24}$$

$$\begin{aligned}
\text{(Dual)}\quad \Gamma(\mathcal{N}) = \min\ & \mu \\
\text{s.t. } & Y_{AB}, V_{AB} \geq 0, \\
& (V_{AB} - Y_{AB})^{T_B} \geq J_{\mathcal{N}}, \\
& \operatorname{Tr}_B (V_{AB} + Y_{AB}) \leq \mu \mathbb{1}_A.
\end{aligned} \tag{25}$$

Proof: Suppose the optimal solution in the optimization (6) of $Q^{(1)}_{PPT}(\mathcal{N}, \varepsilon)$ is taken at $\{W_{AB}, \rho_A, m\}$, then $Q^{(1)}_{PPT}(\mathcal{N}, \varepsilon) = -\log m$. Denote $R_{AB} = \frac{1}{m} W_{AB}$ and we can verify that $\{R_{AB}, \rho_A\}$ is a feasible solution to the SDP (24). Thus

$$\begin{aligned}
Q_\Gamma(\mathcal{N}) &\geq \log \operatorname{Tr} J_{\mathcal{N}} R_{AB} && (26)\\
&= \log \frac{1}{m} \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq \log \frac{1}{m}(1-\varepsilon) && (27)\\
&= Q^{(1)}_{PPT}(\mathcal{N}, \varepsilon) + \log(1-\varepsilon). && (28)
\end{aligned}$$

This concludes the proof.

The dual SDP can be derived via the Lagrange multiplier method. The main step is to associate a positive-semidefinite Lagrange multiplier for each inequality constraint. To be specific, we introduce $V_{AB}, Y_{AB} \geq 0$ and a real multiplier μ, and obtain the following Lagrangian:

$$\begin{aligned}
&\operatorname{Tr} J_{\mathcal{N}} R_{AB} \\
&\quad + \operatorname{Tr}\left(\rho_A \otimes \mathbb{1}_B - R_{AB}^{T_B}\right) V_{AB} \\
&\quad + \operatorname{Tr}\left(\rho_A \otimes \mathbb{1}_B + R_{AB}^{T_B}\right) Y_{AB} \\
&\quad + \mu(1 - \operatorname{Tr}\rho_A) \\
&= \mu + \operatorname{Tr} R_{AB}\left(J_{\mathcal{N}} - V_{AB}^{T_B} + Y_{AB}^{T_B}\right) \\
&\quad + \operatorname{Tr}\rho_A (\operatorname{Tr}_B V_{AB} + \operatorname{Tr}_B Y_{AB} - \mu \mathbb{1}_A).
\end{aligned} \tag{29}$$

Hence, the dual SDP is to minimize μ subject to

$$V_{AB}, Y_{AB} \geq 0, \tag{30}$$

$$J_{\mathcal{N}} \leq V_{AB}^{T_B} - Y_{AB}^{T_B}, \tag{31}$$

$$\operatorname{Tr}_B (V_{AB} + Y_{AB}) \leq \mu \mathbb{1}_A. \tag{32}$$

■

Proposition 6: For any quantum channel $\mathcal{N}_1$ and $\mathcal{N}_2$, we have

$$Q_\Gamma(\mathcal{N}_1 \otimes \mathcal{N}_2) = Q_\Gamma(\mathcal{N}_1) + Q_\Gamma(\mathcal{N}_2). \tag{33}$$

Proof: We only need to show that $\Gamma(\mathcal{N}_1 \otimes \mathcal{N}_2) = \Gamma(\mathcal{N}_1)\Gamma(\mathcal{N}_2)$. For the primal problem (24), suppose the optimal solutions of the SDP (24) for $\mathcal{N}_1$ and $\mathcal{N}_2$ are $\{R_1, \rho_1\}$ and $\{R_2, \rho_2\}$, respectively. Then we can verify that $\{R_1 \otimes R_2, \rho_1 \otimes \rho_2\}$ is a feasible solution of $\Gamma(\mathcal{N}_1 \otimes \mathcal{N}_2)$. Thus,

$$\begin{aligned}
\Gamma(\mathcal{N}_1 \otimes \mathcal{N}_2) &\geq \operatorname{Tr}\left(J_{\mathcal{N}_1} \otimes J_{\mathcal{N}_2}\right)(R_1 \otimes R_2) && (34)\\
&= \Gamma(\mathcal{N}_1)\Gamma(\mathcal{N}_2). && (35)
\end{aligned}$$

For the dual problem (25), suppose the optimal solutions of the SDP (25) for $\mathcal{N}_1$ and $\mathcal{N}_2$ are $\{V_1, Y_1, \mu_1\}$ and $\{V_2, Y_2, \mu_2\}$. Denote $V = V_1 \otimes V_2 + Y_1 \otimes Y_2$ and $Y = V_1 \otimes Y_2 + Y_1 \otimes V_2$. It can be easily verified that $\{V, Y, \mu_1\mu_2\}$ is a feasible solution of $\Gamma(\mathcal{N}_1 \otimes \mathcal{N}_2)$. Thus,

$$\Gamma(\mathcal{N}_1 \otimes \mathcal{N}_2) \leq \Gamma(\mathcal{N}_1)\Gamma(\mathcal{N}_2). \tag{36}$$

■

Theorem 7: For any quantum channel $\mathcal{N}$, we have

$$Q(\mathcal{N}) \leq Q_{\text{PPT}}(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}). \tag{37}$$

Moreover, $Q_\Gamma(\mathcal{N})$ is a strong converse bound. That is, if the rate exceeds $Q_\Gamma(\mathcal{N})$, the error probability will approach to one exponentially fast as the number of channel uses increase.

Proof: We first show that $Q_\Gamma(\mathcal{N})$ is a converse bound and then prove that it is a strong converse. From Eq. (23), take regularization on both sides, we have

$$\begin{aligned}
Q_{\text{PPT}}(\mathcal{N}) &= \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} Q^{(1)}_{\text{PPT}}\left(\mathcal{N}^{\otimes n}, \varepsilon\right) \\
&\leq \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n}\left[Q_\Gamma\left(\mathcal{N}^{\otimes n}\right) - \log(1-\varepsilon)\right] \\
&= Q_\Gamma(\mathcal{N}).
\end{aligned} \tag{38}$$

In the last line, we use the additivity of Q_Γ in Proposition 6.

For the quantum channel $\mathcal{N}^{\otimes n}$, suppose its achievable rate is r. From Eq. (23), we know that $nr \leq nQ_\Gamma(\mathcal{N}) - \log(1-\varepsilon)$, which implies

$$\varepsilon \geq 1 - 2^{n(Q_\Gamma(\mathcal{N}) - r)}. \tag{39}$$

If $r > Q_\Gamma(\mathcal{N})$, the error will exponentially converge to one as n increases. ■

Remark For d-dimensional noiseless quantum channel $\mathcal{I}_d$, we can show $Q(\mathcal{I}_d) = Q_\Gamma(\mathcal{I}_d) = \log d$.

B. Comparison With Other Converse Bounds

There are several well-known converse bounds on quantum capacity. In this subsection, we compare them with our SDP strong converse bound Q_Γ.

Tomamichel et al. [24] established that the Rains information of any quantum channel is a strong converse rate for quantum communication. To be specific, the Rains information of a quantum channel is defined as [24]:

$$R(\mathcal{N}) := \max_{\rho_A \in \mathcal{S}(A)} \min_{\sigma_{AB} \in \text{PPT}'} D\left(\mathcal{N}_{A' \to B}(\phi_{AA'}) \,\middle\|\, \sigma_{AB}\right), \tag{40}$$

where $\phi_{AA'}$ is a purification of ρ_A and the set PPT' $= \{\sigma \in \mathcal{P}(A \otimes B) : \|\sigma^{T_B}\|_1 \leq 1\}$. We note that our bound Q_Γ is weaker than the Rains information (cf. Corollary 10). However, $R(\mathcal{N})$ is not known to be efficiently computable for general quantum channels since it is a max-min optimization problem.

An efficiently computable converse bound (abbreviated as ε-DEG) is given by the concept of approximate degradable channel [14]. This bound usually works very well for approximate degradable quantum channels such as low-noise qubit depolarizing channel. See [50] and [51] for some recent works based on this approach. Otherwise, it will degenerate to a trivial upper bound. We can easily show an example that Q_Γ can be smaller than ε-DEG bound, e.g., the channel $\mathcal{N}_r$ in Eq. (62) with $0 < r < 0.38$. Also, it is unknown whether ε-DEG bound is a strong converse.

Another previously known efficiently computable strong converse bound for general channels is given by the partial transposition bound [25], [26],

$$Q_\Theta(\mathcal{N}) := \log \|\mathcal{N} \circ T\|_\diamond, \tag{41}$$

where T is the transpose map and $\|\cdot\|_\diamond$ is the completely bounded trace norm. Note that which $\|\cdot\|_\diamond$ is known to be efficiently computable via semidefinite programming in [52].

The entanglement cost of a quantum channel [53], denoted as E_C, is proved to be a strong converse bound. But it is not known to be efficiently computable for general channels, due to its regularization. The entanglement-assisted quantum Q_E is also a strong converse for the quantum capacity [54], [55] and there is a recently developed approach to efficiently compute it [56]. Quantum capacity with symmetric side channels [13], denoted as Q_{ss}, is also an important converse bound for general channels. But it is not known to be computable due to the potentially unbounded dimension of the side channel. It is also not known to be a strong converse.

Theorem 8: For any quantum channel $\mathcal{N}$, we have

$$Q(\mathcal{N}) \leq R(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}) \leq Q_\Theta(\mathcal{N}). \tag{42}$$

The first inequality has been proved in [24]. We prove the second inequality in Corollary 10 and the third inequality in Proposition 11.

In the following proof, we need to introduce an entanglement measure E_W which is defined in [41]. We will see that the strong converse bound Q_Γ is a channel analogue of entanglement measure E_W and can be further reformulated into a similar form as the Rains information. Specifically, for any bipartite quantum state ρ_{AB}, the entanglement measure E_W is defined by $E_W(\rho) := \log W(\rho)$, where

$$\text{(Primal)} \quad W(\rho) = \max\left\{\operatorname{Tr}\rho R_{AB} : \left|R_{AB}^{T_B}\right| \leq \mathbb{1}, R_{AB} \geq 0\right\}, \tag{43}$$

$$\text{(Dual)} \quad W(\rho) = \min\left\{\left\|X_{AB}^{T_B}\right\|_1 : X_{AB} \geq \rho_{AB}\right\}. \tag{44}$$

The max-relative entropy of two operators $\rho \in \mathcal{S}_\leq(A)$, $\sigma \in \mathcal{P}(A)$ is defined by [57]

$$D_{\max}(\rho\|\sigma) := \log\min\{\mu : \rho \leq \mu\sigma\}. \tag{45}$$

Proposition 9: For any quantum channel $\mathcal{N}$, it holds that

$$\begin{aligned} Q_\Gamma(\mathcal{N}) &= \max_{\rho_A \in \mathcal{S}(A)} E_W(\mathcal{N}_{A'\to B}(\phi_{AA'})) &(46)\\ &= \max_{\rho\in S(A)} \min_{\sigma\in\text{PPT}'} D_{\max}\left(\mathcal{N}_{A'\to B}(\phi_{AA'}) \|\sigma_{AB}\right), &(47)\end{aligned}$$

where $\phi_{AA'}$ is a purification of ρ_A and the set PPT' $= \left\{\sigma \in \mathcal{P}(A \otimes B) : \|\sigma^{T_B}\|_1 \leq 1\right\}$.

Proof: Consider purification $\phi_{AA'} = \rho_A^{1/2}\Phi_{AA'}\rho_A^{1/2}\left(= \rho_{A'}^{1/2}\Phi_{AA'}\rho_{A'}^{1/2}\right)$, then

$$\begin{aligned} \mathcal{N}_{A'\to B}(\phi_{AA'}) &= \mathcal{N}_{A'\to B}\left(\rho_A^{1/2}\Phi_{AA'}\rho_A^{1/2}\right) &(48)\\ &= \rho_A^{1/2}\mathcal{N}_{A'\to B}(\Phi_{AA'})\rho_A^{1/2} &(49)\\ &= \rho_A^{1/2} J_\mathcal{N} \rho_A^{1/2}. &(50)\end{aligned}$$

Take $J_\mathcal{N} = \rho_A^{-1/2}\mathcal{N}_{A'\to B}(\phi_{AA'})\rho_A^{-1/2}$ into the definition of $Q_\Gamma(\mathcal{N})$ (24) and substitute $F_{AB} = \rho_A^{-1/2}R_{AB}\rho_A^{-1/2}$, we have

$$\begin{aligned} Q_\Gamma(\mathcal{N}) = \log\max &\operatorname{Tr}\mathcal{N}_{A'\to B}(\phi_{AA'}) F_{AB}\\ \text{s.t. } &F_{AB}, \rho_A \geq 0, \operatorname{Tr}\rho_A = 1,\\ &-\mathbb{1}_{AB} \leq F_{AB}^{T_B} \leq \mathbb{1}_{AB} \end{aligned} \tag{51}$$

Note that here we only consider invertible state ρ_A. The reason is that the set of invertible positive operators is dense in the set of all positive semidefinite operators, and then it suffices to optimize with respect to them.

Due to the definition of E_W (43), we have

$$Q_\Gamma(\mathcal{N}) = \max_{\rho_A \in \mathcal{S}(A)} E_W(\mathcal{N}_{A'\to B}(\phi_{AA'})). \tag{52}$$

On the other hand, the following equality chain holds

$$\begin{aligned} E_W(\rho) &= \log\min\left\{\left\|X^{T_B}\right\|_1 : \rho \leq X\right\}\\ &= \log\min\left\{\mu : \rho \leq X, \left\|X^{T_B}\right\|_1 \leq \mu\right\}\\ &= \log\min\left\{\mu : \rho \leq \mu\sigma, \left\|\mu\sigma^{T_B}\right\|_1 \leq \mu\right\}\\ &= \log\min\left\{\mu : \rho \leq \mu\sigma, \left\|\sigma^{T_B}\right\|_1 \leq 1\right\}\\ &= \min_{\sigma\in\text{PPT}'} D_{\max}(\rho\|\sigma). \end{aligned} \tag{53}$$

The first line follows from Eq. (44). In the second line, we introduce a new variable μ. In the third line, we substitute X with $\mu\sigma$. The last line follows from the definition of $D_{\max}$. This directly implies that $E_W(\rho) \geq R(\rho)$ (note, also).[1]

Therefore, we have that

$$\begin{aligned} Q_\Gamma(\mathcal{N}) &= \max_{\rho_A\in\mathcal{S}(A)} E_W(\mathcal{N}_{A'\to B}(\phi_{AA'})) &(54)\\ &= \max_{\rho\in S(A)} \min_{\sigma\in\text{PPT}'} D_{\max}\left(\mathcal{N}_{A'\to B}(\phi_{A'A}) \|\sigma_{AB}\right). &(55)\end{aligned}$$

■

We note that the max-relative entropy of entanglement of a quantum channel [58] and our bound Q_Γ are in the same spirit, but for evaluating quantum communication, the bound in [58] is weaker than the bound Q_Γ as well as the Rains information [24].

[1] We note that Andreas Winter told us the fact that E_W can be proved to be an upper bound of the Rains bound by some optimization techniques in the past.

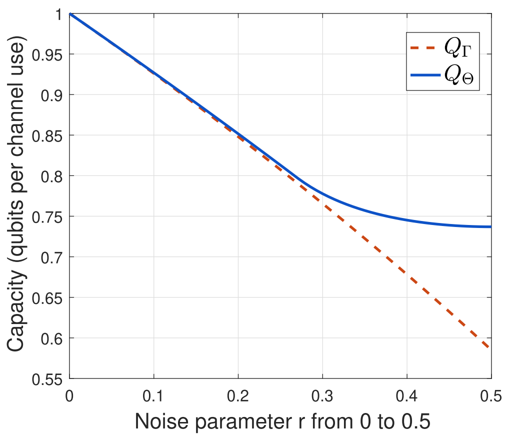

Fig. 4. This plot demonstrates the difference between converse bounds $Q_\Gamma(\mathcal{N}_r)$ and $Q_\Theta(\mathcal{N}_r)$. The dashed line depicts $Q_\Gamma(\mathcal{N}_r)$ while the solid line depicts $Q_\Theta(\mathcal{N}_r)$. The parameter r ranges from 0 to 0.5.

Corollary 10: For any quantum channel $\mathcal{N}$, it holds that

$$R(\mathcal{N}) \leq Q_\Gamma(\mathcal{N}). \tag{56}$$

Proof: Note that $D(\rho\|\sigma) \leq D_{\max}(\rho\|\sigma)$ [57], we have

$$\begin{aligned} Q_\Gamma(\mathcal{N}) &= \max_{\rho\in S(A)} \min_{\sigma\in \mathrm{PPT}'} D_{\max}\left(\mathcal{N}_{A'\to B}(\phi_{A'A}) \|\sigma_{AB}\right) \\ &\geq \max_{\rho_A\in \mathcal{S}(A)} \min_{\sigma\in \mathrm{PPT}'} D\left(\mathcal{N}_{A'\to B}(\phi_{AA'}) \|\sigma_{AB}\right) \\ &= R(\mathcal{N}). \end{aligned} \tag{57}$$

■

Proposition 11: For any quantum channel $\mathcal{N}$, it holds that $Q_\Gamma(\mathcal{N}) \leq Q_\Theta(\mathcal{N})$.

Proof: Suppose the optimal solution of SDP (24) is taken at $\{R_{AB}, \rho_A\}$, then $\Gamma(\mathcal{N}) = \mathrm{Tr}\, J_\mathcal{N} R_{AB} = \mathrm{Tr}\, J_\mathcal{N}^{T_B} R_{AB}^{T_B}$. The completely bounded trace norm can be written as SDP [52],

$$\begin{aligned} \|\mathcal{N}\circ T\|_\diamond = \max\ & \frac{1}{2}\mathrm{Tr}\, J_\mathcal{N}^{T_B}\left(X + X^\dagger\right) \\ \text{s.t.}\ & \begin{pmatrix} \rho_0\otimes \mathbb{1} & X \\ X^\dagger & \rho_1\otimes\mathbb{1}\end{pmatrix} \geq 0, \\ & \rho_0,\rho_1 \in \mathcal{S}(A). \end{aligned} \tag{58}$$

Since $-\rho_A\otimes\mathbb{1}_B \leq R_{AB}^{T_B} \leq \rho_A\otimes\mathbb{1}_B$, we have

$$\begin{aligned} &\begin{pmatrix} \rho_A\otimes\mathbb{1}_B & R_{AB}^{T_B} \\ R_{AB}^{T_B} & \rho_A\otimes\mathbb{1}_B\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \bigotimes \left(\rho_A\otimes\mathbb{1} + R_{AB}^{T_B}\right) \\ &\quad + \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \bigotimes \left(\rho_A\otimes\mathbb{1} - R_{AB}^{T_B}\right) \\ &\geq 0. \end{aligned} \tag{59}$$

So $\left\{R_{AB}^{T_B}, \rho_A, \rho_A\right\}$ is a feasible solution of SDP (58), which means that

$$Q_\Theta(\mathcal{N}) = \log\|\mathcal{N}\circ T\|_\diamond \geq \log \mathrm{Tr}\left(J_\mathcal{N}^{T_B} R_{AB}^{T_B}\right) \tag{60}$$

$$= \log\Gamma(\mathcal{N}) = Q_\Gamma(\mathcal{N}). \tag{61}$$

■

In Fig. 4, we compare the converse bound Q_Γ with Q_Θ in the case of quantum channel

$$\mathcal{N}_r = \sum_{i=0}^{1} E_i\cdot E_i^\dagger, \tag{62}$$

where $E_0 = |0\rangle\langle 0| + \sqrt{r}|1\rangle\langle 1|$ and $E_1 = \sqrt{1-r}|0\rangle\langle 1| + |1\rangle\langle 2|$ $(0\leq r\leq 0.5)$. In the following Fig. 4, it is clear that $Q_\Gamma(\mathcal{N})$ can be strictly tighter than $Q_\Theta(\mathcal{N})$.

V. Discussions

In summary, we have derived efficiently computable converse bounds to evaluate the capabilities of quantum communication over quantum channels in both the non-asymptotic and asymptotic settings by utilizing the techniques of convex optimization.

We have provided one-shot converse bounds in the context of quantum communication with finite resources, which improves the previous general SDP converse bound in [40]. Furthermore, in the asymptotic regime, we have derived an SDP strong converse bound Q_Γ for quantum communication, which is better than the partial transpose bound [25]. Furthermore, we have refined the Q_Γ as the so-called max-Rains information via connecting it to the SDP entanglement measure in [41]. It is worth noting that our bound is no better than the Rains information [24] in general, but it is the best SDP-computable strong converse bound. It is also worth noting that our bound Q_Γ was recently proved to be a strong converse bound for the LOCC-assisted quantum capacity in [59].

However, for the qubit depolarizing channel, the bound Q_Γ does not work very well. The best to date converse bound of this particular channel is still given by [14], [15], and [31]. It is of great interest to use the one-shot SDP converse bound in Eq. (11) to provide a potentially better upper bound on the quantum capacity of depolarizing channel.

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