Semidefinite programming converse bounds for quantum communication

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We study the one-shot and asymptotic quantum communication assisted with the positive-partial-transpose-preserving (PPT) and no-signalling (NS) codes. We first show improved general semidefinite programming (SDP) finite blocklength converse bounds for quantum communication with a given infidelity tolerance and utilize them to study the depolarizing channel and amplitude damping channel in a small blocklength. Based on the one-shot bounds, we then derive a general SDP strong converse bound for the quantum capacity of an arbitrary quantum channel. In particular, we prove that the SDP strong converse bound is always smaller than or equal to the *partial transposition bound* introduced by Holevo and Werner, and the inequality could be strict. Furthermore, we show that the SDP strong converse bound can be refined as the *max-Rains information*, which is an analog to the Rains information introduced in [Tomamichel/Wilde/Winter, *IEEE Trans. Inf. Theory* **63**:715, 2017]. This also implies that it is always no smaller than the Rains information. Finally, we establish an inequality relationship among some of these known strong converse bounds on quantum capacity.

I. INTRODUCTION

A. Background

The reliable transmission of quantum information via noisy quantum channels is a fundamental problem in quantum information theory. The quantum capacity of a noisy quantum channel is the optimal rate at which it can convey quantum bits (qubits) reliably over asymptotically many uses of the channel. The theorem by Lloyd, Shor, and Devetak (LSD) [2–4] and the work in Refs. [5–7] show that the quantum capacity is equal to the regularized coherent information. The quantum capacity is notoriously difficult to evaluate since it is characterized by a multi-letter, regularized expression. Our understanding of the quantum capacity remains limited since it is not even known to be computable [8] and the capacity of basic channels (e.g., depolarizing channel) is still unsolved.

The converse part of the LSD theorem states that if the rate exceeds the quantum capacity, then the fidelity of any coding scheme cannot approach one in the limit of many channel uses. A strong converse property leaves no room for the trade-off between rate and error, i.e., the error probability vanishes in the limit of many channel uses whenever the rate exceeds the capacity. For classical channels, Wolfowitz [9] established the the strong converse property for the classical capacity. For quantum channels, the strong converse property for the classical capacity is confirmed for several classes of channels [10–15].

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For quantum communication, the strong converse property was studied in Ref. [16] and such property of generalized dephasing channels was established [16]. Given an arbitrary quantum channel, a previously known efficiently computable strong converse bound on the quantum capacity for general channels is the partial transposition bound [17], which was proved to be a strong converse bound for the two-way assisted quantum capacity [18]. Recently, the Rains information [16] was established to be a strong converse bound for quantum communication. For the setting of weak converse, there are other known upper bounds for quantum capacity [19–27] and most of them require specific settings to be computable and relatively tight.

Moreover, in a practical setting, the number of quantum channel uses is finite and one has to make a trade-off between the transmission rate and error tolerance. For both practical and theoretical interest, it is important to optimize the trade-off the rate and infidelity of quantum communication with a finite blocklength. The study of this finite blocklength setting has recently attracted great interest in classical information theory (e.g., [28, 29]) as well as in quantum information theory (e.g., [30–42]).

B. Summary of results

In this paper, we focus on the quantum communication via noisy quantum channels in both one-shot and asymptotic settings. We will study the quantum capacity assisted with positive partial transpose preserving (PPT) and no-signalling (NS) codes [34]. The PPT codes include all the operations that can be implemented by local operations and classical communication while the NS codes are potentially stronger than entanglement-assisted codes.

In section III, we consider the non-asymptotic quantum capacity. We first introduce the one-shot ε-infidelity quantum capacity with PPT (and NS) codes and characterize it as an optimization problem. Based on this optimization, we provide a hierarchy of SDPs evaluate the one-shot capacity with a given infidelity tolerance. Comparing with the previous efficiently computable converse bound given in Ref. [40], we show that our SDP converse bounds are tighter in general and can be strictly tighter for basic channels such as the qubit amplitude damping channel and the qubit depolarizing channel.

In section IV, we investigate the asymptotic scenario. We first present an SDP strong converse bound, denoted as Q_Γ, on the quantum capacity for general channels. For any code with a rate exceeding Q_Γ, the infidelity of quantum communication goes to one exponentially fast in the limit of many channel uses. This converse bound has some nice properties, such as additivity under tensor product. In particular, we show that Q_Γ is a channel analog of SDP entanglement measure E_W [43] and can be further refined into a similar optimization form as the Rains information [16] in the sense of replacing the relative entropy with the max-relative entropy. This result implies that Q_Γ is always no smaller than the Rains information. We also remark that in the case of entanglement breaking channels with non-zero classical capacity, Q_Γ can be strictly tighter than the entanglement-assisted quantum capacity. Finally, we show that our Q_Γ is always tighter than the partial transposition bound and can be strictly tighter in some cases.

II. PRELIMINARIES

In the following, we will frequently use symbols such as A (or A') and B (or B') to denote (finite-dimensional) Hilbert spaces associated with Alice and Bob, respectively. We use d_A to denote the dimension of system A. The set of linear operators over A is denoted by $\mathcal{L}(A)$. The set of positive operators over A is denoted by $\mathcal{P}(A)$. The set of positive operators with unit trace is denoted by $\mathcal{S}(A)$, while the set of positive operators with trace no greater than 1 is denoted by $\mathcal{S}_{\leq}(A)$. We usually write an operator with subscript indicating the system that the operator acts on, such as M_{AB}, and write $M_A := \operatorname{Tr}_B M_{AB}$. Note that for a linear operator $R \in \mathcal{L}(A)$, we define $|R| = \sqrt{R^\dagger R}$, where $R^\dagger$ is the adjoint operator of R, and the trace norm of R is given by $\|R\|_1 = \operatorname{Tr}|R|$. A quantum channel $\mathcal{N}_{A'\to B}$ is simply a completely positive (CP) and trace-preserving (TP) linear map from $\mathcal{L}(A')$ to $\mathcal{L}(B)$. The Choi-Jamiołkowski matrix of $\mathcal{N}$ is given by $J_{\mathcal{N}} = \sum_{ij} |i_A\rangle\langle j_A| \otimes \mathcal{N}(|i_{A'}\rangle\langle j_{A'}|)$, where $\{|i_A\rangle\}$ and $\{|i_{A'}\rangle\}$ are orthonormal bases on isomorphic Hilbert spaces A and A', respectively. A positive semidefinite (PSD) operator $E \in \mathcal{L}(A \otimes B)$ is said to be a positive partial transpose operator (or simply PPT) if $E^{T_B} \geq 0$, where T_B means the partial transpose with respect to the party B, i.e., $(|ij\rangle\langle kl|)^{T_B} = |il\rangle\langle kj|$. As shown in Ref. [44], a bipartite operation $\Pi_{A_iB_i\to A_oB_o}$ is PPT-preserving if and only if its Choi-Jamiołkowski matrix $Z_{A_iB_iA_oB_o}$ is PPT.

The constraints of PPT and NS can be mathematically characterized as follows. A bipartite operation $\Pi_{A_iB_i\to A_oB_o}$ is no-signalling and PPT-preserving if and only if its Choi-Jamiołkowski matrix $Z_{A_iB_iA_oB_o}$ satisfies [34]:

$$\begin{aligned} Z_{A_iB_iA_oB_o} &\geq 0, && \text{(CP)} \\ Z_{A_iB_i} &= \mathbb{1}_{A_iB_i}, && \text{(TP)} \\ Z_{A_iB_iA_oB_o}^{T_{B_iB_o}} &\geq 0, && \text{(PPT)} \\ Z_{A_iB_iB_o} &= \frac{\mathbb{1}_{A_i}}{d_{A_i}} \otimes Z_{B_iB_o}, && (A \not\to B) \\ Z_{A_iB_iA_o} &= \frac{\mathbb{1}_{B_i}}{d_{B_i}} \otimes Z_{A_iA_o}, && (B \not\to A) \end{aligned} \tag{1}$$

where the five lines correspond to characterize that Π is CP, TP, PPT, NS from A to B, NS from B to A, respectively. Note that the mathematical structure of quantum no-signalling correlations (or NS codes) was also studied in Ref. [45].

Semidefinite programming (SDP) [46] is a useful tool in the study of quantum information and computation with many applications (e.g., [47–57]). In this work, we use the CVX software [58] and QETLAB (A Matlab Toolbox for Quantum Entanglement) [59] to solve the SDPs.

III. CONVERSE BOUNDS FOR NON-ASYMPTOTIC QUANTUM COMMUNICATION

A. One-shot ε-error capacity and finite resource trade-off

In this section we are interested in the finite blocklength regime of quantum communication and focus on codes enabling a state entangled with a reference system to be reliably transmitted. Suppose Alice shares a maximally entangled state (Φ_{A_iR}) with a reference system R to which she has no access. The goal is to design a quantum coding protocol such that Alice can transfer this maximally entangled state to Bob with as high fidelity as possible. To this end, Alice needs to perform some encoding channel $\mathcal{E}_{A_i\to A_o}$ on system A_i to prepare it for input and then transmits the prepared state $\mathcal{E}_{A_i\to A_o}(\Phi_{A_iR})$ through the channel $\mathcal{N}_{A_o\to B_i}$, resulting in the state $\mathcal{N}_{A_o\to B_i} \circ \mathcal{E}_{A_i\to A_o}(\Phi_{A_iR})$. Once Bob receives the state from the channel output, he performs some decoding channel $\mathcal{D}_{B_i\to B_o}$, where B_o is some system of the same dimension as A_i. The final state after Bob's decoding will be $\mathcal{D}_{B_i\to B_o} \circ \mathcal{N}_{A_o\to B_i} \circ \mathcal{E}_{A_i\to A_o}(\Phi_{A_iR})$. We can also denote encoder $\mathcal{E}_{A_i\to A_o}$ and decoder $\mathcal{D}_{B_i\to B_o}$ as a general superoperator $\Pi_{A_iB_i\to A_oB_o}$. Thus the final state can be written as $\Pi \circ \mathcal{N}(\Phi_{A_iR})$. Note that Π is a bipartite quantum operation form A_iB_i to A_oB_o. Adding different constraints on Π, such as PPT-preserving (PPT) or non-signalling (NS) constraints [34, 45, 60], we

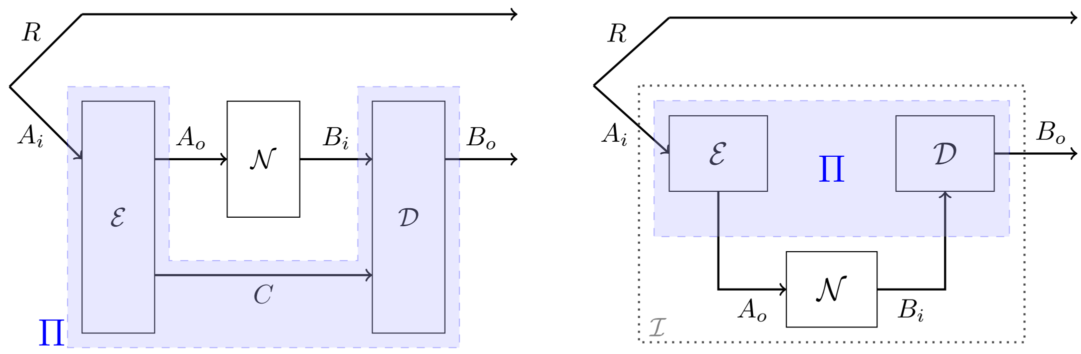

FIG. 1: Bipartite operation $\Pi_{A_iB_i\to A_oB_o}$ is equivalently the coding scheme ($\mathcal{E}$,$\mathcal{D}$) with free extra resources C, such as entanglement or no-signalling correlations. The whole operation is to simulate a noiseless quantum channel $\mathcal{I}_{A_i\to B_o}$ using a given noisy quantum channel $\mathcal{N}_{A_o\to B_i}$ and the bipartite code Π.

will obtain different codes. In the following, Ω denotes specific class of codes, i.e., $\Omega \in \{\mathrm{NS}\cap\mathrm{PPT}, \mathrm{PPT}\}$.

Definition 1 *The maximum channel fidelity of $\mathcal{N}$ assisted by the Ω-class code are defined by*

$$F_\Omega(\mathcal{N},k) := \sup_\Pi \operatorname{Tr}\left(\Phi_{B_oR}\cdot\Pi\circ\mathcal{N}\left(\Phi_{A_iR}\right)\right), \tag{2}$$

where Φ_{A_iR} and Φ_{B_oR} are maximally entangled states, $k = dim|A_i| = dim|B_o|$ called code size and the supremum is taken over all the codes in class Ω.

Definition 2 *For given quantum channel $\mathcal{N}$ and error tolerance ε, the one-shot ε-error quantum capacity assisted by Ω-class codes is defined by*

$$Q_\Omega^{(1)}(\mathcal{N},\varepsilon) := \log\max\left\{k\in\mathbb{N} : F_\Omega(\mathcal{N},k)\geq 1-\varepsilon\right\}. \tag{3}$$

The asymptotic quantum capacity is then given by

$$Q_\Omega(\mathcal{N}) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}Q_\Omega^{(1)}\left(\mathcal{N}^{\otimes n},\varepsilon\right). \tag{4}$$

Considering PPT (and NS) codes, the maximum channel fidelity is then given by SDP [34],

$$\begin{aligned} F_\Omega(\mathcal{N},k) = \max\ & \operatorname{Tr} J_{\mathcal{N}}W_{AB} \\ \text{s.t. } & 0\leq W_{AB}\leq\rho_A\otimes\mathbb{1}_B, \operatorname{Tr}\rho_A = 1, \\ \textbf{PPT:} & -k^{-1}\rho_A\otimes\mathbb{1}_B\leq W_{AB}^{T_B}\leq k^{-1}\rho_A\otimes\mathbb{1}_B, \\ \textbf{NS:} & \operatorname{Tr}_A W_{AB} = k^{-2}\mathbb{1}_B. \end{aligned} \tag{5}$$

Proposition 3 *For any quantum channel $\mathcal{N}_{A'\to B}$ and given error tolerance ε, its one-shot ε-error quantum capacity with PPT codes can be simplified as an optimization problem:*

$$\begin{aligned} Q_{PPT}^{(1)}(\mathcal{N},\varepsilon) = -\log\min\ & m \\ \textit{s.t. } & \operatorname{Tr} J_{\mathcal{N}}W_{AB}\geq 1-\varepsilon, 0\leq W_{AB}\leq\rho_A\otimes\mathbb{1}_B, \\ & \operatorname{Tr}\rho_A = 1, -m\rho_A\otimes\mathbb{1}_B\leq W_{AB}^{T_B}\leq m\rho_A\otimes\mathbb{1}_B. \end{aligned} \tag{6}$$

If the codes are also non-signalling, we can have the same optimization for $Q_{PPT\cap NS}^{(1)}(\mathcal{N},\varepsilon)$ with additional constraint $\operatorname{Tr}_A W_{AB} = m^2\mathbb{1}_B$.

Proof This result can be easily proved by combining Eq. (3) and (5). It is worth noting that Eq. (6) is not an SDP in general, due to the non-linear term $m\rho_A$ and the condition $\operatorname{Tr}_A W_{AB} = m^2 \mathbb{1}_B$. But in the following discussions, we will have several methods to relax them to semidefinite conditions. □

B. Improved SDP converse bounds for quantum communication

To better evaluate the quantum communication rate with finite resources, we introduce some SDP converse bounds for quantum communication with the assistance of PPT (and NS) codes. We then prove in Theorem 4 that our SDP bounds are tighter than the one introduced in Ref. [40]. Examples have been given in the next subsection to show that our bounds can be strictly tighter.

Specifically, the authors in Ref. [40] show that $-\log f(\mathcal{N}, \varepsilon)$ is a converse bound on one-shot ε-error quantum capacity, i.e., $Q^{(1)}(\mathcal{N}, \varepsilon) \leq -\log f(\mathcal{N}, \varepsilon)$ where

$$\begin{aligned} f(\mathcal{N}, \varepsilon) = \min \operatorname{Tr} S_A & \\ \text{s.t. } & \operatorname{Tr} W_{AB} J_{\mathcal{N}} \geq 1-\varepsilon, S_A, \Theta_{AB} \geq 0, \operatorname{Tr} \rho_A = 1, \\ & 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, S_A \otimes \mathbb{1}_B \geq W_{AB} + \Theta_{AB}^{T_B}. \end{aligned} \tag{7}$$

Here, we introduce a hierarchy of SDP converse bounds on the one-shot ε-error capacity based on the optimization (6). If we relax the term $m\rho_A$ to a single variable S_A, we obtain $g(\mathcal{N}, \varepsilon)$, where

$$\begin{aligned} g(\mathcal{N}, \varepsilon) := \min \operatorname{Tr} S_A & \\ \text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr} \rho_A = 1, -S_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B} \leq S_A \otimes \mathbb{1}_B. \end{aligned} \tag{8}$$

In particular, if we further consider the NS condition $\operatorname{Tr}_A W_{AB} = m^2 \mathbb{1}_B$, we can have two different relaxations. The first one is to substitute it with $\operatorname{Tr}_A W_{AB} = t\mathbb{1}_B$ and get the SDP $\widetilde{g}(\mathcal{N}, \varepsilon)$ while the second method is to introduce a prior constant $\widehat{m}$ satisfying the inequality

$$Q^{(1)}_{PPT \cap NS}(\mathcal{N}, \varepsilon) \leq -\log \widehat{m} \tag{9}$$

and get the SDP $\widehat{g}(\mathcal{N}, \varepsilon)$. Note that the second method can provide a tighter bound, but it requires one more step of calculation since we need to give the prior constant $\widehat{m}$. Successively refining the value of $\widehat{m}$ will result in a tighter bound.

$$\begin{aligned} \widetilde{g}(\mathcal{N}, \varepsilon) := \min \operatorname{Tr} S_A & \\ \text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr} \rho_A = 1, -S_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B} \leq S_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}_A W_{AB} = t\mathbb{1}_B. \end{aligned} \tag{10}$$

$$\begin{aligned} \widehat{g}(\mathcal{N}, \varepsilon) := \min \operatorname{Tr} S_A & \\ \text{s.t. } & \operatorname{Tr} J_{\mathcal{N}} W_{AB} \geq 1-\varepsilon, 0 \leq W_{AB} \leq \rho_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr} \rho_A = 1, -S_A \otimes \mathbb{1}_B \leq W_{AB}^{T_B} \leq S_A \otimes \mathbb{1}_B, \\ & \operatorname{Tr}_A W_{AB} = t\mathbb{1}_B, t \geq \widehat{m}^2. \end{aligned} \tag{11}$$

Theorem 4 *For any quantum channel $\mathcal{N}$ and error tolerance ε, the inequality chain holds*

$$Q^{(1)}(\mathcal{N}, \varepsilon) \leq Q^{(1)}_{PPT \cap NS}(\mathcal{N}, \varepsilon) \leq -\log \widehat{g}(\mathcal{N}, \varepsilon) \leq -\log \widetilde{g}(\mathcal{N}, \varepsilon) \leq -\log g(\mathcal{N}, \varepsilon) \leq -\log f(\mathcal{N}, \varepsilon). \tag{12}$$

Proof The first inequality is trivial. The third and fourth inequalities are also easy to obtain since minimizing over a smaller feasible set gives a larger optimal value.

For the second inequality, suppose the optimal solution of (6) for $Q^{(1)}_{PPT\cap NS}(\mathcal{N},\varepsilon)$, is taken at $\{W_{AB},\rho_A,m\}$. Let $S_A = m\rho_A,\ t = m^2$. Then we can verify that $\{W_{AB},\rho_A,S_A,t\}$ is a feasible solution to the SDP (11) of $\widehat{g}(\mathcal{N},\varepsilon)$. So $\widehat{g}(\mathcal{N},\varepsilon) \le \operatorname{Tr} S_A = m$, which implies $Q^{(1)}_{PPT\cap NS}(\mathcal{N},\varepsilon) = -\log m \le -\log \widehat{g}(\mathcal{N},\varepsilon)$.

For the last inequality, we only need to show that $f(\mathcal{N},\varepsilon) \le g(\mathcal{N},\varepsilon)$. Suppose the optimal solution of $g(\mathcal{N},\varepsilon)$ is taken at $\{\rho_A,S_A,W_{AB}\}$. Let us choose $\Theta_{AB} = S_A\otimes \mathbb{1}_B - W_{AB}^{T_B}$. Since $S_A\otimes\mathbb{1}_B \ge W_{AB}^{T_B}$, it is clear that $\Theta_{AB}\ge 0$ and $S_A\otimes\mathbb{1}_B = W_{AB}+\Theta_{AB}^{T_B}$. Thus, $\{S_A,\rho_A,W_{AB},\Theta_{AB}\}$ is a feasible solution to the SDP (7) of $f(\mathcal{N},\varepsilon)$ which implies $f(\mathcal{N},\varepsilon)\le \operatorname{Tr} S_A = g(\mathcal{N},\varepsilon)$. □

C. Examples: amplitude damping channel and depolarizing channel

In this subsection, we study the examples of amplitude damping channel and depolarizing channel. We show in Fig. 2 that for the amplitude damping channel $\mathcal{N}_{AD}$, our converse bound $-\log\widetilde{g}(\mathcal{N},\varepsilon)$ and $-\log g(\mathcal{N},\varepsilon)$ are both tighter than $-\log f(\mathcal{N},\varepsilon)$. For the depolarizing channel $\mathcal{N}_D$, exploiting its symmetry, we can further simplify its SDPs into linear programs. Thus converse bounds $-\log f(\mathcal{N}^{\otimes n},\varepsilon)$, $-\log g(\mathcal{N}^{\otimes n},\varepsilon)$, $-\log\widetilde{g}(\mathcal{N}^{\otimes n},\varepsilon)$, $-\log\widehat{g}(\mathcal{N}^{\otimes n},\varepsilon)$ can be easily calculated for the n-fold tensor product depolarizing channel, $\mathcal{N}_D^{\otimes n}$. We show in Fig. 3 that the converse bound $-\log\widehat{g}(\mathcal{N}^{\otimes n},\varepsilon)$ can be strictly tighter than $-\log g(\mathcal{N}^{\otimes n},\varepsilon)$ after a few times of successive refinement of the value $\widehat{m}$.

Example For the amplitude damping channel $\mathcal{N}_{AD} = \sum_{i=0}^{1} E_i\cdot E_i^\dagger$ with $E_0 = |0\rangle\langle 0| + \sqrt{1-r}|1\rangle\langle 1|$, $E_1 = \sqrt{r}|0\rangle\langle 1|$ $(0\le r\le 1)$, the differences among $-\log f\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right)$, $-\log g\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right)$ and $-\log\widetilde{g}\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right)$, are presented in Fig. 2. When $r\in(0.082,0.094)$, $-\log\widetilde{g}\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right) \le -\log g\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right) < 1 < -\log f\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right)$. It shows that we cannot transmit a single qubit within error tolerance $\varepsilon = 0.01$ via 2 copies of amplitude damping channel where parameter $r\in(0.082,0.094)$. However, this result is not indicated by the converse bound $-\log f\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right)$. □

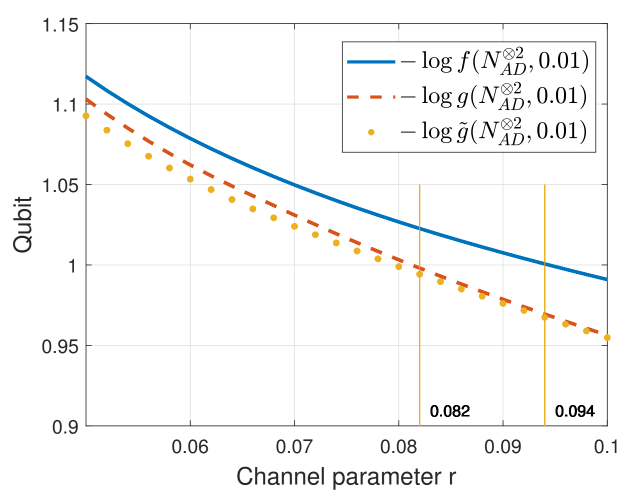

FIG. 2: This figure demonstrates the differences among the SDP converse bounds (i) $-\log f\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right)$ (blue solid), (ii) $-\log g\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right)$ (red dashed), (iii) $-\log\widetilde{g}\left(\mathcal{N}_{AD}^{\otimes 2},0.01\right)$ (yellow dotted), where the channel parameter r ranges from 0.05 to 0.1.

Example For the qubit depolarizing channel $\mathcal{N}_D(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$, where X, Y, Z are Pauli matrices, the Choi matrix of $\mathcal{N}_D$ is $J_{\mathcal{N}} = d\left((1-p)\Phi + \frac{p}{d^2-1}\Phi^{\perp}\right)$, where $d = 2$, $\Phi = \frac{1}{d}\sum_{i,j=0}^{d-1}|ii\rangle\langle jj|$ and $\Phi^{\perp} = \mathbb{1}_{AB} - \Phi$. For the n-fold tensor product depolarizing channel, its Choi matrix is $J_{\mathcal{N}}^{\otimes n} = d^n \sum_{i=0}^{n} f_i P_i^n(\Phi, \Phi^{\perp})$, where $f_i = (1-p)^i\left(\frac{p}{d^2-1}\right)^{n-i}$ and $P_i^n(\Phi,\Phi^{\perp})$ represent the sum of those n-fold tensor product terms with exactly i copies of Φ. For example,

$$P_1^3\left(\Phi, \Phi^{\perp}\right) = \Phi^{\perp}\otimes\Phi^{\perp}\otimes\Phi + \Phi^{\perp}\otimes\Phi\otimes\Phi^{\perp} + \Phi\otimes\Phi^{\perp}\otimes\Phi^{\perp}. \tag{13}$$

Suppose $\{W_{AB}, \rho_A, S_A\}$ is the optimal solution to the SDP (8) for the channel $\mathcal{N}_D^{\otimes n}$, then for any local unitary $U = \otimes_{i=1}^{n} U_A^i \otimes \overline{U}_B^i$, $U_A = \otimes_{i=1}^{n} U_A^i$, we know that $\{UWU^{\dagger}, U_A\rho_A U_A^{\dagger}, U_A S_A U_A^{\dagger}\}$ is also optimal. Convex combinations of optimal solutions remain optimal. Without loss of generality, we can take the optimal solution to be invariant under any local unitary U and U_A, respectively. Again, since $J_{\mathcal{N}}^{\otimes n}$ is invariant under the symmetric group, acting by permuting the tensor factors. We can finally take the optimal solution as $W = \sum_{i=0}^{n} w_i P_i^n(\Phi, \Phi^{\perp})$, $\rho_A = \mathbb{1}_A/d^n$, $S_A = s\mathbb{1}_A$.

Note that $P_i^n(\Phi,\Phi^{\perp})$ are orthogonal projections. Thus without considering degeneracy, operator W has eigenvalues $\{w_i\}_{i=0}^{n}$. Next, we need to know the eigenvalues of W^{T_B}. Decomposing operators Φ^{T_B} and $\Phi^{\perp T_B}$ into orthogonal projections, i.e.,

$$\Phi^{T_B} = \frac{1}{d}(P_+ - P_-), \quad \Phi^{\perp T_B} = \left(1 - \frac{1}{d}\right)P_+ + \left(1 + \frac{1}{d}\right)P_- \tag{14}$$

where P_+ and P_- are symmetric and anti-symmetric projections respectively and collecting the terms with respect to $P_k^n(P_+, P_-)$, we have

$$W^{T_B} = \sum_{i=0}^{n} w_i P_i^n\left(\Phi^{T_B}, \Phi^{\perp T_B}\right) = \sum_{k=0}^{n}\left(\sum_{i=0}^{n} x_{i,k} w_i\right) P_k^n(P_+, P_-), \quad \text{where} \tag{15}$$

$$x_{i,k} = \frac{1}{d^n}\sum_{m=\max\{0,i+k-n\}}^{\min\{i,k\}} \binom{k}{m}\binom{n-k}{i-m}(-1)^{i-m}(d-1)^{k-m}(d+1)^{n-k+m-i}. \tag{16}$$

Since $P_k^n(P_+, P_-)$ are also orthogonal projections, W^{T_B} has eigenvalues $\{t_k\}_{k=0}^{n}$ (without considering degeneracy), where $t_k = \sum_{i=0}^{n} x_{i,k} w_i$. As for the constraint $\operatorname{Tr} J_{\mathcal{N}}^{\otimes n} W_{AB} \geq 1 - \varepsilon$, we have

$$\operatorname{Tr} J_{\mathcal{N}}^{\otimes n} W = d^n \operatorname{Tr}\sum_{i=0}^{n} f_i w_i P_i^n\left(\Phi, \Phi^{\perp}\right) = d^n \sum_{i=0}^{n}\binom{n}{i}(1-p)^i p^{n-i} w_i \geq 1 - \varepsilon. \tag{17}$$

Finally, substitute $\eta = sd^n$ and $m_i = w_i d^n$. We obtain the linear program

$$\begin{aligned} g\left(\mathcal{N}_D^{\otimes n}, \varepsilon\right) = \min\ & \eta \\ \text{s.t.} & \sum_{i=0}^{n}\binom{n}{i}(1-p)^i p^{n-i} m_i \geq 1 - \varepsilon, \\ & 0 \leq m_i \leq 1,\ i = 0, 1, \cdots, n, \\ & -\eta \leq \sum_{i=0}^{n} x_{i,k} m_i \leq \eta,\ k = 0, 1, \cdots, n. \end{aligned} \tag{18}$$

Following a similar procedure, we have

$$
\begin{aligned}
f\left(\mathcal{N}_D^{\otimes n},\varepsilon\right) = \min\ & \eta \\
\text{s.t. } & \sum_{i=0}^{n}\binom{n}{i}(1-p)^i p^{n-i} m_i \geq 1-\varepsilon, \\
& m_i + s_i \leq \eta,\ i=0,1,\cdots,n, \\
& \eta \geq 0,\ 0\leq m_i \leq 1,\ i=0,1,\cdots,n \\
& \sum_{i=0}^{n} x_{i,k} s_i \geq 0,\ k=0,1,\cdots,n.
\end{aligned}
$$

$$
\begin{aligned}
\widehat{g}\left(\mathcal{N}_D^{\otimes n},\varepsilon\right) = \min\ & \eta \\
\text{s.t. } & \sum_{i=0}^{n}\binom{n}{i}(1-p)^i p^{n-i} m_i \geq 1-\varepsilon, \\
& 0\leq m_i \leq 1,\ i=0,1,\cdots,n, \\
& -\eta \leq \sum_{i=0}^{n} x_{i,k} m_i \leq \eta,\ k=0,1,\cdots,n, \\
& \frac{1}{d^{2n}}\sum_{i=0}^{n}\binom{n}{i}\left(d^2-1\right)^{n-i} m_i \geq \widehat{m}^2.
\end{aligned}
$$

Since $-\log\widehat{g}\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$ is a converse bound for any $\widehat{m} \leq 2^{-Q^{(1)}_{\text{PPT}\cap\text{NS}}\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)}$, we can successively refine the value of $\widehat{m}$ and obtain a tighter result. Denote $\widehat{m}_i$ and $\widehat{g}_i\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$ the value of $\widehat{m}$ and $\widehat{g}\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$ in the i-th iteration. First, we take initial value of $\widehat{m}_1 = g\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$ and get the result $\widehat{g}_1\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$. Then set $\widehat{m}_{i+1} = \widehat{g}_i\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$ and get result $\widehat{g}_{i+1}\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$. In Fig. 3, we show that after five iterations, we can get a converse bound $-\log\widehat{g}_5\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$ strictly tighter than $-\log f\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$. Especially, when $n=17$, $-\log\widehat{g}_5\left(\mathcal{N}_D^{\otimes n},\varepsilon\right) < 1 < -\log f\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$. It shows that we cannot transmit a single qubit within error tolerance $\varepsilon = 0.004$ via 17 copies of depolarizing channel where parameter $p=0.2$. However, this result is not indicated by the converse bound $-\log f\left(\mathcal{N}_D^{\otimes n},\varepsilon\right)$. □

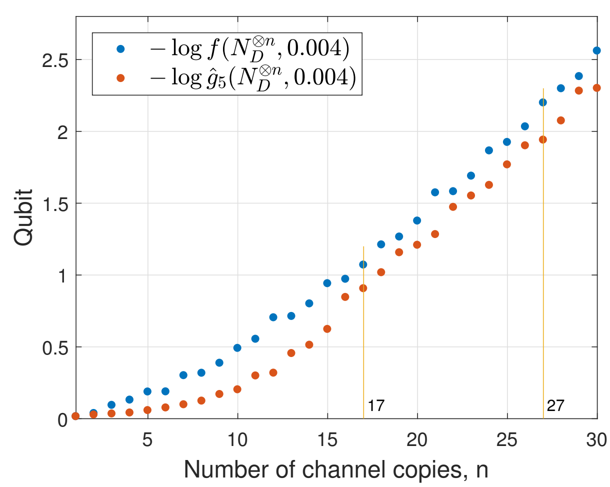

FIG. 3: This figure demonstrates the differences between the SDP converse bounds $-\log f\left(\mathcal{N}_D^{\otimes n}, 0.004\right)$ (blue dots) and $-\log\widehat{g}_5\left(\mathcal{N}_D^{\otimes n}, 0.004\right)$ (red dots), where the channel parameter $p=0.2$ and the number of channel uses ranges from 1 to 30.

IV. STRONG CONVERSE BOUND FOR QUANTUM COMMUNICATION

In this section, we introduce an SDP strong converse bound $Q_\Gamma(\mathcal{N})$ to evaluate the quantum capacity for general quantum channels. We summarize our strong converse bound with other well-known bounds in Tab. I. Among those efficiently computable strong converse bound for general channels, we prove that $Q_\Gamma(\mathcal{N})$ is better than the partial transpose bound and remark that it is also strictly tighter than the entanglement-assisted quantum capacity in the case of entanglement-breaking channels with non-zero classical capacity. The relation with Rains information is also obtained.

A. An SDP strong converse bound on quantum capacity

Proposition 5 *For any quantum channel $\mathcal{N}$ and error tolerance ε,*

$$Q_{PPT}^{(1)}(\mathcal{N},\varepsilon) \le Q_\Gamma(\mathcal{N}) - \log(1-\varepsilon), \tag{19}$$

where $Q_\Gamma(\mathcal{N}) := \log\Gamma(\mathcal{N})$ and

$$\textbf{(Primal)}\quad \Gamma(\mathcal{N}) = \max\left\{\operatorname{Tr} J_\mathcal{N} R_{AB} : R_{AB}, \rho_A \ge 0, \operatorname{Tr}\rho_A = 1, -\rho_A\otimes\mathbb{1}_B \le R_{AB}^{T_B} \le \rho_A\otimes\mathbb{1}_B.\right\} \tag{20}$$

$$\textbf{(Dual)}\quad \Gamma(\mathcal{N}) = \min\left\{\mu : Y_{AB}, V_{AB} \ge 0, (V_{AB}-Y_{AB})^{T_B} \ge J_\mathcal{N}, \operatorname{Tr}_B(V_{AB}+Y_{AB}) \le \mu\mathbb{1}_A.\right\} \tag{21}$$

Proof Suppose the optimal solution in the optimization (6) of $Q_{PPT}^{(1)}(\mathcal{N},\varepsilon)$ is taken at $\{W_{AB},\rho_A,m\}$, then $Q_{PPT}^{(1)}(\mathcal{N},\varepsilon) = -\log m$. Denote $R_{AB} = \frac{1}{m}W_{AB}$ and we can verify that $\{R_{AB},\rho_A\}$ is a feasible solution to the SDP (20). Thus

$$Q_\Gamma(\mathcal{N}) \ge \log\operatorname{Tr} J_\mathcal{N} R_{AB} = \log\frac{1}{m}\operatorname{Tr} J_\mathcal{N} W_{AB} \ge \log\frac{1}{m}(1-\varepsilon) = Q_{PPT}^{(1)}(\mathcal{N},\varepsilon) + \log(1-\varepsilon).$$

This concludes the proof. The dual problem can be derived via Lagrange multiplier method. □

Proposition 6 *For any quantum channel $\mathcal{N}_1$ and $\mathcal{N}_2$, Q_Γ is additive, i.e.,*

$$Q_\Gamma(\mathcal{N}_1\otimes\mathcal{N}_2) = Q_\Gamma(\mathcal{N}_1) + Q_\Gamma(\mathcal{N}_2). \tag{22}$$

Proof We only need to show that $\Gamma(\mathcal{N}_1\otimes\mathcal{N}_2) = \Gamma(\mathcal{N}_1)\Gamma(\mathcal{N}_2)$. For the primal problem (20), suppose the optimal solutions of (20) for the channel $\mathcal{N}_1$ and $\mathcal{N}_2$ are taken at $\{R_1,\rho_1\}$ and $\{R_2,\rho_2\}$, respectively. Then we can verify that $\{R_1\otimes R_2,\rho_1\otimes\rho_2\}$ is a feasible solution of $\Gamma(\mathcal{N}_1\otimes\mathcal{N}_2)$. Thus $\Gamma(\mathcal{N}_1\otimes\mathcal{N}_2) \ge \operatorname{Tr}(J_{\mathcal{N}_1}\otimes J_{\mathcal{N}_2})(R_1\otimes R_2) = \Gamma(\mathcal{N}_1)\Gamma(\mathcal{N}_2)$.

For the dual problem (21), suppose the optimal solutions of (21) for the channel $\mathcal{N}_1$ and $\mathcal{N}_2$ are taken at $\{V_1,Y_1,\mu_1\}$ and $\{V_2,Y_2,\mu_2\}$. Denote $V = V_1\otimes V_2 + Y_1\otimes Y_2$ and $Y = V_1\otimes Y_2 + Y_1\otimes V_2$. Then we can verify that $\{V,Y,\mu_1\mu_2\}$ is a feasible solution of $\Gamma(\mathcal{N}_1\otimes\mathcal{N}_2)$. Thus $\Gamma(\mathcal{N}_1\otimes\mathcal{N}_2) \le \Gamma(\mathcal{N}_1)\Gamma(\mathcal{N}_2)$. □

Theorem 7 *For any quantum channel $\mathcal{N}$, $Q_\Gamma(\mathcal{N})$ is a converse bound on PPT-asssited quantum capacity,*

$$Q(\mathcal{N}) \le Q_{PPT}(\mathcal{N}) \le Q_\Gamma(\mathcal{N}). \tag{23}$$

Moreover, $Q_\Gamma(\mathcal{N})$ is a strong converse bound. That is, if the rate exceeds $Q_\Gamma(\mathcal{N})$, the error probability will approach to one exponentially fast as the number of channel uses increase.

Proof We first show that $Q_\Gamma(\mathcal{N})$ is a converse bound and then prove that it is a strong converse. From Eq. (19), take regularization on both sides, we have

$$\begin{aligned} Q_{PPT}(\mathcal{N}) &= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}Q_{PPT}^{(1)}\left(\mathcal{N}^{\otimes n},\varepsilon\right) \\ &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\left[Q_\Gamma\left(\mathcal{N}^{\otimes n}\right) - \log(1-\varepsilon)\right] \\ &= Q_\Gamma(\mathcal{N}). \end{aligned} \tag{24}$$

In the last line, we use the additivity of Q_Γ in Proposition 6.

For the n-fold quantum channel $\mathcal{N}^{\otimes n}$, suppose its achievable rate is r. From Eq. (19), we have $nr \leq nQ_\Gamma(\mathcal{N}) - \log(1-\varepsilon)$, which implies

$$\varepsilon \geq 1 - 2^{n(Q_\Gamma(\mathcal{N})-r)}. \tag{25}$$

If $r > Q_\Gamma(\mathcal{N})$, the error will exponentially converge to one as n goes to infinity. □

Remark For d-dimensional noiseless quantum channel $\mathcal{I}_d$, we can show $Q(\mathcal{I}_d) = Q_\Gamma(\mathcal{I}_d) = \log d$.

B. Comparison with other converse bounds

There are several well-known converse bounds on quantum capacity. In this subsection, we compare them with our SDP strong converse bound Q_Γ. Especially, we obtain an inequality chain among the strong converse bound Q_Γ, channel's Rains information R and partial transposition bound Q_Θ.

	Strong converse	Efficiently computable	For general channels
Q_Γ	✓	✓	✓
R	✓	? (max-min)	✓
ε-DEG	?	✓	✗
E_C	✓	? (regularization)	✓
Q_E	✓	✓	✓
Q_{ss}	?	? (unbounded dimension)	✓
Q_Θ	✓	✓	✓

TABLE I: Comparison of converse bounds on quantum capacity. The check mark represents that the property holds while the cross mark represents that the property does not hold. The question mark represents the unknown result. The words in the bracket explain the difficulty that stops us to make it computable. The shaded rows indicate the bounds we particularly discuss in the following part.

The channel's Rains information, denoted as R, is proved to be a strong converse bound on quantum capacity. However, it is not known to be efficiently computable for general quantum channels due to its max-min optimization form.

$$R(\mathcal{N}) := \max_{\rho_A \in \mathcal{S}(A)} \min_{\sigma \in \mathrm{PPT}'} D\left(\mathcal{N}_{A'\to B}(\phi_{AA'}) \| \sigma\right), \tag{26}$$

where $\phi_{AA'}$ is a purification of ρ_A and the set $\mathrm{PPT}' = \{\sigma \in \mathcal{P}(A\otimes B) : \|\sigma^{T_B}\|_1 \leq 1\}$.

An efficiently computable converse bound (abbreviated as ε-DEG) is given by the concept of approximate degradable channel [20]. This bound usually works very well for approximate degradable quantum channels such as low-noise qubit depolarizing channel. See Ref. [61**?**] for some recent works based on this approach. Otherwise, it will degenerate to a trivial upper bound. We can easily show an example that Q_Γ can be smaller than ε-DEG bound, e.g., the channel $\mathcal{N}_r$ in Eq. (42) with $0 < r < 0.38$. Also, it is unknown whether ε-DEG bound is a strong converse.

The entanglement cost of a quantum channel [62], denoted as E_C, is proved to be a strong converse bound. But it is not known to be efficiently computable for general channels, due to its regularization.

Entanglement-assisted quantum capacity, denoted as Q_E, is also a strong converse for the unassisted quantum capacity [32, 63]. Moreover, it holds that $Q_E(\mathcal{N}) = \frac{1}{2}C_E(\mathcal{N})$, where C_E is the entanglement-assisted classical capacity which is efficiently computable [64].

Quantum capacity with symmetric side channels [19], denoted as Q_{ss}, is also an important converse bound for general channels. But it is not known to be computable due to the potentially unbounded dimension of the side channel. It is also not known to be a strong converse.

Another previously known efficiently computable strong converse bound for general channels is given by the partial transposition bound,

$$Q_{\Theta}(\mathcal{N}) := \log \|\mathcal{N} \circ T\|_{\diamond}, \tag{27}$$

where T is transpose map and $\|\cdot\|_{\diamond}$ is the completely bounded trace norm, which is known to be efficiently computable by SDP in Ref. [65].

Theorem 8 *For any quantum channel $\mathcal{N}$, it holds*

$$Q(\mathcal{N}) \leq R(\mathcal{N}) \leq Q_{\Gamma}(\mathcal{N}) \leq Q_{\Theta}(\mathcal{N}). \tag{28}$$

The first inequality has been proved in Ref. [16]. We prove the second inequality in Corollary 10 and the third inequality in Proposition 11.

In the following proof, we need to introduce an entanglement measure E_W which is defined in Ref. [43]. We will see that the strong converse bound Q_{Γ} is a channel analogue of entanglement measure E_W and can be further reformulated into a similar form as the Rains information. Specifically, for any bipartite quantum state ρ_{AB}, $E_W(\rho) := \log W(\rho)$ where

$$\textbf{(Primal)} \quad W(\rho) = \max\left\{\operatorname{Tr} \rho R_{AB} : \left|R_{AB}^{T_B}\right| \leq \mathbb{1}, R_{AB} \geq 0\right\}, \tag{29}$$

$$\textbf{(Dual)} \quad W(\rho) = \min\left\{\left\|X_{AB}^{T_B}\right\|_1 : X_{AB} \geq \rho_{AB}\right\}. \tag{30}$$

The max-relative entropy of two operators $\rho \in \mathcal{S}_{\leq}(A)$, $\sigma \in \mathcal{P}(A)$ is defined by [66]

$$D_{\max}(\rho\|\sigma) := \log \min\{\mu : \rho \leq \mu\sigma\}. \tag{31}$$

Proposition 9 *For any quantum channel $\mathcal{N}$, it holds*

$$Q_{\Gamma}(\mathcal{N}) = \max_{\rho_A \in \mathcal{S}(A)} E_W(\mathcal{N}_{A' \to B}(\phi_{AA'})) = \max_{\rho \in S(A)} \min_{\sigma \in \mathrm{PPT}'} D_{\max}\left(\mathcal{N}_{A' \to B}(\phi_{AA'}) \|\sigma_{AB}\right), \tag{32}$$

where $\phi_{AA'}$ is a purification of ρ_A and the set PPT′ $= \left\{\sigma \in \mathcal{P}(A \otimes B) : \left\|\sigma^{T_B}\right\|_1 \leq 1\right\}$.

Proof Consider purification $\phi_{AA'} = \rho_A^{1/2} \Phi_{AA'} \rho_A^{1/2} \left(= \rho_{A'}^{1/2} \Phi_{AA'} \rho_{A'}^{1/2}\right)$, then

$$\mathcal{N}_{A' \to B}(\phi_{AA'}) = \mathcal{N}_{A' \to B}\left(\rho_A^{1/2} \Phi_{AA'} \rho_A^{1/2}\right) = \rho_A^{1/2} \mathcal{N}_{A' \to B}(\Phi_{AA'}) \rho_A^{1/2} = \rho_A^{1/2} J_{\mathcal{N}} \rho_A^{1/2}. \tag{33}$$

Take $J_{\mathcal{N}} = \rho_A^{-1/2} \mathcal{N}_{A' \to B}(\phi_{AA'}) \rho_A^{-1/2}$ into the definition of $Q_{\Gamma}(\mathcal{N})$ (20) and substitute $F_{AB} = \rho_A^{-1/2} R_{AB} \rho_A^{-1/2}$, we have

$$\begin{aligned} Q_{\Gamma}(\mathcal{N}) = \log \max \operatorname{Tr} \mathcal{N}_{A' \to B}(\phi_{AA'}) F_{AB} \\ \text{s.t. } F_{AB}, \rho_A \geq 0, \operatorname{Tr} \rho_A = 1, -\mathbb{1}_{AB} \leq F_{AB}^{T_B} \leq \mathbb{1}_{AB} \end{aligned} \tag{34}$$

Due to the definition of E_W (29), we have

$$Q_{\Gamma}(\mathcal{N}) = \max_{\rho_A \in \mathcal{S}(A)} E_W(\mathcal{N}_{A' \to B}(\phi_{AA'})). \tag{35}$$

On the other hand, the following equality chain holds

$$
\begin{aligned}
E_W(\rho) &= \log \min \left\{ \left\| X^{T_B} \right\|_1 : \rho \le X \right\} \\
&= \log \min \left\{ \mu : \rho \le X, \left\| X^{T_B} \right\|_1 \le \mu \right\} \\
&= \log \min \left\{ \mu : \rho \le \mu\sigma, \left\| \mu\sigma^{T_B} \right\|_1 \le \mu \right\} \\
&= \log \min \left\{ \mu : \rho \le \mu\sigma, \left\| \sigma^{T_B} \right\|_1 \le 1 \right\} \\
&= \min_{\sigma \in \mathrm{PPT}'} D_{\max}(\rho \| \sigma).
\end{aligned} \tag{36}
$$

The first line follows from Eq. (30). In the second line, we introduce a new variable μ. In the third line, we substitute X with $\mu\sigma$. The last line follows from the definition of $D_{\max}$. This directly implies that $E_W(\rho) \ge R(\rho)$. We also note that Andreas Winter [67] told us the fact that E_W can be proved to be an upper bound of the Rains bound by some optimization techniques in the past.

Therefore,

$$
Q_\Gamma(\mathcal{N}) = \max_{\rho_A \in \mathcal{S}(A)} E_W\left(\mathcal{N}_{A' \to B}(\phi_{AA'})\right) = \max_{\rho \in S(A)} \min_{\sigma \in \mathrm{PPT}'} D_{\max}\left(\mathcal{N}_{A' \to B}(\phi_{A'A}) \| \sigma_{AB}\right). \tag{37}
$$

□

Remark From this proposition, it is clear that $Q_\Gamma(\mathcal{N})$ vanishes for any entanglement breaking channel, since any output state $\mathcal{N}_{A' \to B}(\phi_{AA'})$ is separable and $E_W(\mathcal{N}_{A' \to B}(\phi_{AA'})) = 0$. Thus for any entanglement breaking channel $\mathcal{N}$ with non-zero classical capacity, we have $Q_E(\mathcal{N}) = \frac{1}{2} C_E(\mathcal{N}) \ge \frac{1}{2} C(\mathcal{N}) > 0 = Q_\Gamma(\mathcal{N})$.

Corollary 10 *For any quantum channel $\mathcal{N}$, it holds $R(\mathcal{N}) \le Q_\Gamma(\mathcal{N})$.*

Proof Note that $D(\rho\|\sigma) \le D_{\max}(\rho\|\sigma)$ [66], we have

$$
\begin{aligned}
Q_\Gamma(\mathcal{N}) &= \max_{\rho \in S(A)} \min_{\sigma \in \mathrm{PPT}'} D_{\max}\left(\mathcal{N}_{A' \to B}(\phi_{A'A}) \| \sigma_{AB}\right) \\
&\ge \max_{\rho_A \in \mathcal{S}(A)} \min_{\sigma \in \mathrm{PPT}'} D\left(\mathcal{N}_{A' \to B}(\phi_{AA'}) \| \sigma_{AB}\right) = R(\mathcal{N}).
\end{aligned} \tag{38}
$$

□

Proposition 11 *For any quantum channel $\mathcal{N}$, it holds $Q_\Gamma(\mathcal{N}) \le Q_\Theta(\mathcal{N})$.*

Proof Suppose the optimal solution of SDP (20) is taken at $\{R_{AB}, \rho_A\}$, then $\Gamma(\mathcal{N}) = \mathrm{Tr}\, J_{\mathcal{N}} R_{AB} = \mathrm{Tr}\, J_{\mathcal{N}}^{T_B} R_{AB}^{T_B}$. The completely bounded trace norm can be written as SDP [65],

$$
\|\mathcal{N} \circ T\|_\diamond = \max \left\{ \frac{1}{2} \mathrm{Tr}\, J_{\mathcal{N}}^{T_B} \left(X + X^\dagger\right) : \begin{pmatrix} \rho_0 \otimes \mathbb{1} & X \\ X^\dagger & \rho_1 \otimes \mathbb{1} \end{pmatrix} \ge 0,\ \rho_0, \rho_1 \in \mathcal{S}(A). \right\} \tag{39}
$$

Since $-\rho_A \otimes \mathbb{1}_B \le R_{AB}^{T_B} \le \rho_A \otimes \mathbb{1}_B$, we have

$$
\begin{pmatrix} \rho_A \otimes \mathbb{1}_B & R_{AB}^{T_B} \\ R_{AB}^{T_B} & \rho_A \otimes \mathbb{1}_B \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \bigotimes \left(\rho_A \otimes \mathbb{1} + R_{AB}^{T_B}\right) + \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \bigotimes \left(\rho_A \otimes \mathbb{1} - R_{AB}^{T_B}\right) \ge 0. \tag{40}
$$

So $\left\{R_{AB}^{T_B}, \rho_A, \rho_A\right\}$ is a feasible solution of SDP (39), which means that

$$
Q_\Theta(\mathcal{N}) = \log \|\mathcal{N} \circ T\|_\diamond \ge \log \mathrm{Tr}\left(J_{\mathcal{N}}^{T_B} R_{AB}^{T_B}\right) = \log \Gamma(\mathcal{N}) = Q_\Gamma(\mathcal{N}). \tag{41}
$$

□

In Fig. 4, we compare the converse bound Q_Γ with Q_Θ in the case of quantum channel

$$\mathcal{N}_r = \sum_{i=0}^{1} E_i \cdot E_i^\dagger, \tag{42}$$

where $E_0 = |0\rangle\langle 0| + \sqrt{r}|1\rangle\langle 1|$ and $E_1 = \sqrt{1-r}|0\rangle\langle 1| + |1\rangle\langle 2|$ $(0 \le r \le 0.5)$. In the following Fig. 4, it is clear that $Q_\Gamma(\mathcal{N})$ can be strictly tighter than $Q_\Theta(\mathcal{N})$.

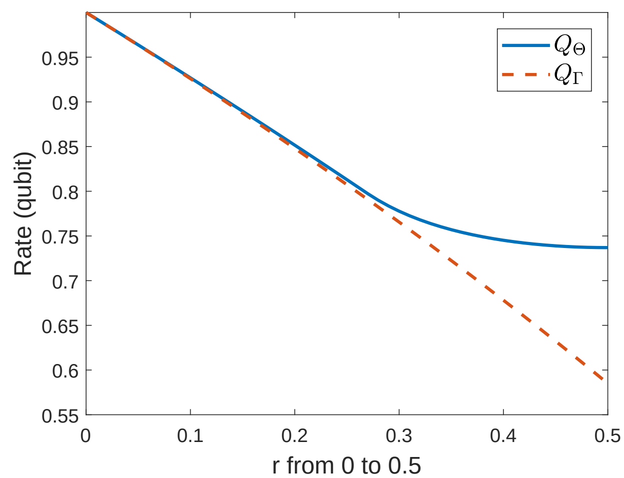

FIG. 4: This plot demonstrates the difference between converse bounds $Q_\Gamma(\mathcal{N}_r)$ and $Q_\Theta(\mathcal{N}_r)$. The dashed line depicts $Q_\Gamma(\mathcal{N}_r)$ while the solid line depicts $Q_\Theta(\mathcal{N}_r)$. The parameter r ranges from 0 to 0.5.

V. DISCUSSIONS

In summary, we have derived efficiently computable converse bounds to estimate the capability of quantum communication in both non-asymptotic and asymptotic settings by utilizing the techniques of convex optimization.

We have introduced a hierarchy of SDP converse bounds for the one-shot ε-infidelity quantum capacity, which improves the previous general SDP converse bound in Ref. [40]. In particular, we have shown our SDP converse bounds could be strictly better by applying them to some basic quantum channels such as qubit amplitude damping channel and qubit depolarizing channel. Furthermore, in the asymptotic setting of quantum communication, we have derived an SDP strong converse bound for the quantum capacity and compare it with other well-known converse bounds. In particular, we have proved that our strong converse bound Q_Γ is always tighter than or equal to the partial transpose bound [17]. Furthermore, we have refined the SDP strong converse bound in the form of max-Rains information by connecting it to the SDP entanglement measure in [43]. Finally, we have established an inequality relationship among the known strong converse bounds on quantum capacity,

$$Q(\mathcal{N}) \le R(\mathcal{N}) \le Q_\Gamma(\mathcal{N}) \le Q_\Theta(\mathcal{N}). \tag{43}$$

However, for the qubit depolarizing channel, the bound Q_Γ does not work very well. The best to date converse bound of this particular channel is still given by Refs. [20, 25, 27]. It is of great interest to use the one-shot SDP converse bound in Eq. (11) to provide a potentially better upper bound on the quantum capacity of depolarizing channel. Another interesting problem is to determine the asymptotic quantum capacity assisted by PPT (and NS) codes via the optimization in Proposition 3.

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