

Finite Block Length Analysis on Quantum Coherence Distillation and Incoherent Randomness Extraction

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February 27, 2020

Abstract

We introduce a variant of randomness extraction framework in the context of quantum coherence theory where free incoherent operations are allowed before the incoherent measurement and the randomness extractors. This cryptographic framework opens a new perspective to the study of quantum coherence distillation by an exact one-shot relation, that is, the maximum number of random bits extractable from a given quantum state is precisely equal to the maximum number of coherent bits that can be distilled from the same state. This relation enables us to derive tight second order expansions of both tasks in the independent and identically distributed setting. Remarkably, the incoherent operation classes that can empower coherence distillation for generic states all admit the same second order expansions, indicating their operational equivalence for coherence distillation in both asymptotic and large block length regimes. As a by-product, we showcase an alternative proof of the strong converse property of coherence distillation and randomness extraction from their second order expansions.

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1 Introduction

Quantum coherence is a physical resource that is essential for various tasks in quantum computing (e.g. Deutsch-Jozsa algorithm [Hil16]), cryptography (e.g. quantum key distribution [CML16]), information processing (e.g. quantum state merging [SCR⁺16], state redistribution [AJS18] and channel simulation [DFW⁺18]), thermodynamics [LJR15] and metrology [FD11]. A series of efforts have been devoted to building a resource framework of coherence in recent years [Abe06, GS08, LM14, BCP14, SAP17], characterizing in particular the state transformations and operational uses of coherence in fundamental resource manipulation protocols [WY16, CSR⁺16, SCR⁺16, CH16, RFWA18, FWL⁺18, ZLY⁺18]. As in any physical resource theory, a central problem of the resource theory of quantum coherence is *distillation*: the process of extracting canonical units of coherence, named coherent bits, from a given quantum state using a choice of free operations.

The usual asymptotic approach to studying the problem in quantum information theory is to assume that there is an unbounded number of independent and identically distributed (i.i.d.) copies of a quantum state available and that the transformation error asymptotically goes to zero. The task of coherence distillation in the asymptotic scenario has been first investigated in [WY16] and has been recently completed in [Lam19]. In spite of their theoretical importance, the asymptotic assumptions become unphysical in reality due to our limited access to a finite number of copies of a given state, making it necessary to look at non-asymptotic regimes. The first step in this direction is to consider the one-shot setting that distills coherence from a single instance of the prepared state. Such a scenario has been investigated in [RFWA18] and has been mostly completed in [ZLY⁺18]. These works estimate the one-shot distillable coherence under different free operations by their corresponding one-shot entropies. The one-shot entropies most accurately describe the operational quantity, yet they tend to be difficult to calculate for large systems, even in the i.i.d. case. This motivates us to further investigate the second order expansions of distillable coherence.

More precisely, take the coherence distillation under incoherent operations (IO) as an example. Let $C_{d,\text{IO}}^\varepsilon(\rho)$ be the maximum number of coherent bits that can be distilled from the given state ρ while the transformation error is within ε . The asymptotic result by Winter & Yang [WY16] together with its strong converse property [ZLY⁺19] can be expressed as:

$$C_{d,\text{IO}}^\varepsilon(\rho^{\otimes n}) = nD(\rho\|\Delta(\rho)) + o(n), \quad \text{for all } \varepsilon \in (0, 1), \quad (1)$$

where $D(\cdot\|\cdot)$ is the quantum relative entropy and Δ is the completely dephasing channel in a given basis. The coefficient of n in (1), i.e. $D(\rho\|\Delta(\rho))$, is called the *first order coefficient* of distillable coherence. In a second order expansion, one seeks to further analyze the term $o(n)$, which usually turns out to be of the order \sqrt{n} . One of our key results here is to refine (1) to

$$C_{d,\text{IO}}^\varepsilon(\rho^{\otimes n}) = nD(\rho\|\Delta(\rho)) + \sqrt{nV(\rho\|\Delta(\rho))} \Phi^{-1}(\varepsilon^2) + O(\log n), \quad (2)$$

where $V(\cdot\|\cdot)$ denotes the quantum information variance and Φ^{-1} denotes the inverse of the cumulative distribution function of a standard normal random variable. The coefficient of \sqrt{n} in (2), i.e. $\sqrt{V(\rho\|\Delta(\rho))} \Phi^{-1}(\varepsilon^2)$, is called the *second order coefficient* of the distillable coherence. In general, an expansion into terms in n , \sqrt{n} and $o(\sqrt{n})$ such as (2) is called a *second order expansion*.

The significance of second order expansions is multifold. First, second order expansions of distillable coherence provide a useful approximation for finite block length n , refining optimal rates that typically correspond to the first order coefficient in asymptotic expansions. Second, they determine the rate of convergence of the averaged distillable coherence to its first order coefficient. This is analogous to the relation between the Central Limit Theorem and the Berry-Esseen Theorem, as the latter determines the rate of convergence in the former. Finally, second order expansions of distillable coherence can be used to derive the strong converse property, an information-theoretic property that rules out a possible tradeoff between the transformation error and the distillable coherence of a protocol.

The usual approach to deriving the second order expansion of an information task is to combine the one-shot entropy bounds on the information quantity and the second order expansion of the corresponding entropies (e.g. [Hay08, TH13, Li14, DL14, TBR16, FWTD19, WFT19]). However, as second order expansions have a strong dependence on the error parameter ε , the existing one-shot entropy bounds on distillable coherence [RFWA18, ZLY⁺19] are insufficient to get a tight second order expansion. That is, the second order coefficients in the expansion of the one-shot entropy lower and upper bounds are often mismatched. To solve

this, we introduce a variant of randomness extraction framework in the context of quantum coherence theory [YZCM15, HZ18] and build an exact connection of this task with coherence distillation. Such a connection provides us a new perspective to the study of distillation process. Finally, expanding a one-shot entropy lower bound on the extractable randomness and a one-shot entropy upper bound on the distillable coherence, we obtain the desired second order expansion as presented in (2).

The exact one-shot relation between randomness extraction and coherence distillation builds a bridge between two seemingly different information tasks, providing new perspectives to the study of both problems. Moreover, our second order expansions initiate *the first* large block length analysis in quantum coherence theory, filling an important gap in the literature.

Outline and main contributions The main contributions of this paper can be summarized as follows:

- In Section 3, we propose a variant of randomness extraction framework in the context of quantum coherence theory, which is later proved to be closely related to the task of quantum coherence distillation.
- In Section 4, we establish an *exact* relation between the task of randomness extraction and the task of quantum coherence distillation in the one-shot regime. More precisely, we show that the maximum number of secure randomness bits ($\ell_{\mathcal{O}}^{\varepsilon}$) extractable from a given state is equal to the maximum number of coherent bits ($C_{d,\mathcal{O}}^{\varepsilon}$) that can be distilled from the same state. That is, for any quantum state ρ , error tolerance $\varepsilon \in [0, 1]$, it holds

$$C_{d,\mathcal{O}}^{\varepsilon}(\rho) = \ell_{\mathcal{O}}^{\varepsilon}(\rho), \quad (3)$$

where free operation class $\mathcal{O} \in \{\text{MIO}, \text{DIO}, \text{IO}, \text{DIIO}\}$ whose definitions can be found in Section 3.1.

- In Section 5, we combine the derived one-shot relation between $C_{d,\mathcal{O}}^{\varepsilon}$ and $\ell_{\mathcal{O}}^{\varepsilon}$, their known one-shot entropy bounds as well as the second order expansion of these entropies to get the second order expansion of our information tasks. That is, for any quantum state ρ , error tolerance $\varepsilon \in (0, 1)$, free operation class $\mathcal{O} \in \{\text{MIO}, \text{DIO}, \text{IO}, \text{DIIO}\}$, it holds

$$C_{d,\mathcal{O}}^{\varepsilon}(\rho^{\otimes n}) = \ell_{\mathcal{O}}^{\varepsilon}(\rho^{\otimes n}) = nD(\rho \parallel \Delta(\rho)) + \sqrt{nV(\rho \parallel \Delta(\rho))} \Phi^{-1}(\varepsilon^2) + O(\log n). \quad (4)$$

- In Section 6, we provide an alternative proof of the strong converse property of quantum coherence distillation and incoherent randomness extraction by using the established second order expansions.

The argument of our results uses various technical statements (e.g. properties of smooth entropies), which are presented in Appendix A. Finally, a refined estimation to the one-shot distillable coherence by quantum hypothesis testing relative entropy is given in Appendix B, which provides an alternative proof of our second order expansions.

2 Preliminaries

In this section we define several quantities and set the notation that will be used throughout this paper. We label different physical systems by capital Latin letters (e.g. A, C, L). We often use these labels as subscripts to guide the reader by indicating which system a mathematical object belongs to. We drop the subscripts when they are evident in the context of an expression (or if we are not talking about a specific system). The corresponding Hilbert spaces of these physical systems are denoted as $\mathcal{H}_A, \mathcal{H}_C, \mathcal{H}_L$ respectively. The corresponding alphabet sets are denoted by the same letters in mathcal font (e.g. $\mathcal{A}, \mathcal{C}, \mathcal{L}$). For example, $\mathcal{A} := \{1, 2, \dots, |A|\}$ where $|A|$ is the dimension of Hilbert space \mathcal{H}_A . Let $\{|a\rangle\}_{a \in \mathcal{A}}$ be the computational basis on Hilbert space \mathcal{H}_A . The set of positive semidefinite operators on \mathcal{H}_A is denoted as $\mathcal{P}(A)$. The set of quantum states, which are positive semidefinite operators with unit trace, on \mathcal{H}_A is denoted as $\mathcal{S}(A)$. Denote the completely mixed state on \mathcal{H}_A as π_A . The identity operator and the identity map are denoted as $\mathbb{1}$ and id respectively. A quantum operation $\Lambda_{A \rightarrow C}$ is a completely positive trace-preserving (CPTP) map from $\mathcal{S}(A)$ to $\mathcal{S}(C)$. All logarithms in this work are taken base two.

For any $\rho, \sigma \in \mathcal{P}$, the purified distance P is defined in terms of the generalized quantum fidelity F as $P(\rho, \sigma) := \sqrt{1 - F(\rho, \sigma)^2}$ with $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1 + \sqrt{(1 - \text{Tr} \rho)(1 - \text{Tr} \sigma)}$ [Tom15]. For any $\rho \in \mathcal{S}$

and $\sigma \in \mathcal{P}$, their quantum hypothesis testing relative entropy is defined as $D_H^\varepsilon(\rho\|\sigma) := -\log \min\{\text{Tr } M\sigma : \text{Tr } M\rho \geq 1-\varepsilon, 0 \leq M \leq \mathbb{1}\}$. The smooth max-relative entropy is defined as $D_{\max}^\varepsilon(\rho\|\sigma) := \min_{P(\tilde{\rho}, \rho) \leq \varepsilon} \inf\{\lambda : \tilde{\rho} \leq 2^\lambda \sigma\}$. The second order expansions of quantum hypothesis testing relative entropy [TH13, Li14] and smooth max-relative entropy [TH13] are, respectively, given by

$$D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)} \Phi^{-1}(\varepsilon) + O(\log n), \quad (5)$$

$$D_{\max}^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = nD(\rho\|\sigma) - \sqrt{nV(\rho\|\sigma)} \Phi^{-1}(\varepsilon^2) + O(\log n), \quad (6)$$

where $D(\rho\|\sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)]$ is the quantum relative entropy, $V(\rho\|\sigma) := \text{Tr}[\rho(\log \rho - \log \sigma)^2] - D(\rho\|\sigma)^2$ is the quantum information variance and Φ^{-1} is the inverse of the cumulative distribution function of a standard normal random variable.

3 Quantum coherence distillation and incoherent randomness extraction

In this section we first review the resource theory of quantum coherence and the operational task of quantum coherence distillation. We then introduce a variant of randomness extraction framework in the context of quantum coherence theory which plays a crucial role in the later discussions.

3.1 Resource theory of quantum coherence

The resource theory of coherence consists of the following ingredients [BCP14]: the set of *free states* and the set of *free operations*, that is, a set of quantum operations that do not generate coherence. The free states, so-called incoherent states, are the quantum states which are diagonal in a given reference orthonormal basis $\{|a\rangle\}_{a \in \mathcal{A}}$. We will use $\Delta_A(\cdot) := \sum_{a \in \mathcal{A}} |a\rangle\langle a| \cdot |a\rangle\langle a|$ to denote the diagonal map (completely dephasing channel) in this basis. Then the set of incoherent states is denoted as $\mathcal{I}(A) := \{\rho \in \mathcal{S}(A) : \rho = \Delta_A(\rho)\}$. For convenience, we will also use the cone of diagonal positive semidefinite matrices, which is denoted as $\mathcal{I}^{**}(A) := \{X \in \mathcal{P}(A) : X = \Delta_A(X)\}$. The maximal resource state on \mathcal{H}_A is the *maximally coherent state* $|\Psi_A\rangle := 1/\sqrt{|A|} \sum_{a=1}^{|A|} |a\rangle$ with dimension $|A|$. Denote its density operator as $\Psi_A := |\Psi_A\rangle\langle\Psi_A|$. The resource theory of coherence is known not to admit a unique physically-motivated choice of allowed free operations [WY16, CG16, MS16, DVS16, SAP17]. The relevant choices of free operations that we will focus on are: *maximally incoherent operations (MIO)* [Abe06], defined to be all operations Λ such that $\Lambda(\rho) \in \mathcal{I}$ for every $\rho \in \mathcal{I}$; *dephasing-covariant incoherent operations (DIO)* [CG16, MS16], which are maps Λ such that $\Delta \circ \Lambda = \Lambda \circ \Delta$; *incoherent operations (IO)* [BCP14], which admit a set of incoherent Kraus operators $\{K_l\}$ such that $K_l \rho K_l^\dagger \in \mathcal{I}^{**}$ for all l and $\rho \in \mathcal{I}$; the intersection of IO and DIO is denoted as $\text{DIIO} := \text{DIO} \cap \text{IO}$ [ZLY⁺19]. Another two classes of free operations commonly studied are *strictly incoherent operations (SIO)* [WY16] and *physically incoherent operations (PIO)* [CG16]. We do not investigate further details of SIO and PIO, as it has been recently shown that quantum coherence is generically undistillable under these two classes [LRA19, Lam19]. Finally, the inclusion relations between free operation classes can be summarized as $\text{DIIO} \subsetneq \text{IO} \subsetneq \text{MIO}$, $\text{DIIO} \subsetneq \text{DIO} \subsetneq \text{MIO}$, while IO and DIO are not contained by each other.

3.2 Framework of quantum coherence distillation

The task of *coherence distillation* aims to transform a given quantum state ρ to a maximally coherent state such that the transformation error is within a given threshold and that the obtained maximally coherent state has dimension as large as possible. More formally, for any free operation class \mathcal{O} , any given state $\rho_A \in \mathcal{S}(A)$ and error tolerance $\varepsilon \in [0, 1]$, the *one-shot distillable coherence* is defined as

$$C_{d, \mathcal{O}}^\varepsilon(\rho_A) := \max \{ \log |C| : P(\Lambda_{A \rightarrow C}(\rho_A), \Psi_C) \leq \varepsilon, \Lambda \in \mathcal{O} \}. \quad (7)$$

Note that some previous works (e.g. [RFWA18, FWL⁺18, ZLY⁺19]) estimate the performance of distillation by $P(\Lambda_{A \rightarrow C}(\rho_A), \Psi_C) \leq \sqrt{\varepsilon}$. Here we use the definition in (7) for convenience.

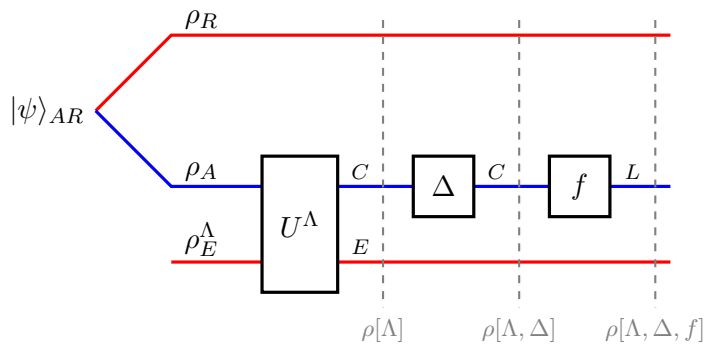


Figure 1: Schematic diagram of an incoherent randomness extraction protocol given by (Λ, Δ, f) . $|\psi\rangle_{AR}$ is a purification of ρ_A . $(\rho_E^\Lambda, U^\Lambda)$ is the Stinespring representation of $\Lambda \in \mathcal{O}$, where ρ_E^Λ is the initial pure state on the environment \mathcal{H}_E and U^Λ is the isometry from $\mathcal{H}_A \otimes \mathcal{H}_E$ to $\mathcal{H}_C \otimes \mathcal{H}_E$. Δ is a completely dephasing channel and f is a hash function from alphabet \mathcal{C} to \mathcal{L} . $\rho[\Lambda]$, $\rho[\Lambda, \Delta]$ and $\rho[\Lambda, \Delta, f]$ are respectively the output states in each step of the protocol. The systems in blue belong to Alice and the systems in red belong to Eve.

3.3 Framework of incoherent randomness extraction

The task of *incoherent randomness extraction* aims to obtain as many random bits as possible at Alice's laboratory that is secure from possible attackers such as Eve. A general *incoherent randomness extraction protocol* is characterized by a triplet (Λ, Δ, f) , where Λ is an incoherent operation in a certain class, Δ is a completely dephasing channel and f is a hash function. A detailed procedure of randomness extraction by (Λ, Δ, f) is depicted in Figure 1. For any given quantum state ρ_A held by Alice, we denote its purification as ¹

$$|\psi\rangle_{AR} := \sum_{a \in \mathcal{A}} \sqrt{p_a} |a\rangle_A |\psi_a\rangle_R \quad \text{with} \quad \text{Tr}_R |\psi\rangle\langle\psi|_{AR} = \rho_A, \quad (8)$$

where R is the reference system held by Eve. Alice first performs a free operation $\Lambda_{A \rightarrow C} \in \mathcal{O}$ on her part of the system. If she uses a quantum operation whose final state is always a specific incoherent state, say the maximally mixed state π_C , the resulting conditional entropy equals $\log |C|$, which increases unlimitedly as $|C|$ increases. To avoid such a trivial advantage for Alice, similar to the study of quantum key distribution [TLGR12, SP00, Hay06] and private capacity in quantum Shannon theory [LWZG09], we assume that the environment system \mathcal{H}_E of the free operation Λ is also controlled by Eve. This is because it is not easy to exclude the possibility that Eve accesses a system that interacts with Alice's operation. Hence Eve has control over two systems \mathcal{H}_R and \mathcal{H}_E in total. To cover such a worst scenario, we consider the Stinespring representation $(\rho_E^\Lambda, U^\Lambda)$ of Λ , where ρ_E^Λ is the initial pure state on the environment and U^Λ is the isometry from $\mathcal{H}_A \otimes \mathcal{H}_E$ to $\mathcal{H}_C \otimes \mathcal{H}_E$.² After the action of Λ , the total output state is a pure state

$$\rho[\Lambda]_{CER} := (U^\Lambda)(|\psi\rangle\langle\psi|_{AR} \otimes \rho_E^\Lambda)(U^\Lambda)^\dagger. \quad (9)$$

Next, Alice performs an incoherent measurement, with respect to the computational basis, on her part of the state. The output state is then denoted as

$$\rho[\Lambda, \Delta]_{CER} := \Delta_C(\rho[\Lambda]_{CER}). \quad (10)$$

Finally, a hash function f is applied on her part of the system to extract the randomness that is secure from Eve. For any deterministic function $f: \mathcal{C} \rightarrow \mathcal{L}$, and any classical-quantum (CQ) state $\rho_{CR} = \sum_{c \in \mathcal{C}} t_c |c\rangle\langle c|_C \otimes \rho_{R|c}$, denote $\rho_{f(C)R} := \sum_{c \in \mathcal{C}} t_c |f(c)\rangle\langle f(c)|_L \otimes \rho_{R|c}$. Then the output state in the final step is given by

$$\rho[\Lambda, \Delta, f]_{LER} := \rho[\Lambda, \Delta]_{f(C)ER}. \quad (11)$$

To quantify the security of randomness in a quantum state ρ_{AR} with respect to the reference system R , we employ the following security measure:

$$d_{sec}(\rho_{AR}|R) := \min_{\sigma_R \in \mathcal{S}(R)} P(\rho_{AR}, \pi_A \otimes \sigma_R). \quad (12)$$

¹ Note that $|\psi_a\rangle$ are not necessarily orthogonal to each other.

² Note that a free operation does not necessarily admit a free dilation [CG16]. Thus U^Λ is not necessarily incoherent though Λ is free.

The *one-shot extractable randomness under given free operation* $\Lambda \in \mathcal{O}$ is defined as

$$\ell_{\Lambda}^{\varepsilon}(\rho_A) := \max_f \{ \log |L| : d_{\text{sec}}(\rho[\Lambda, \Delta, f]_{LER}|ER) \leq \varepsilon \}. \quad (13)$$

The *one-shot extractable randomness under free operation class* \mathcal{O} is then defined as

$$\ell_{\mathcal{O}}^{\varepsilon}(\rho_A) := \max_{\Lambda \in \mathcal{O}} \ell_{\Lambda}^{\varepsilon}(\rho_A). \quad (14)$$

Note that the identity map id is always free in coherence theory. Thus (id, Δ, f) is a valid incoherent randomness extraction protocol for any f , which was studied in [TH13, Section III]. That is, Alice directly performs an incoherent measurement Δ on her given state ρ_A . In this case, the environment system \mathcal{H}_E reduces to trivial and system $\mathcal{H}_C = \mathcal{H}_A$. It has been shown in [TH13, Theorem 8] that for any $\eta \in (0, \varepsilon]$,

$$H_{\min}^{\varepsilon-\eta}(A|R)_{\tilde{\rho}} + 4 \log \eta - 3 \leq \ell_{\text{id}}^{\varepsilon}(\rho_A) \leq H_{\min}^{\varepsilon}(A|R)_{\tilde{\rho}}, \quad (15)$$

where $H_{\min}^{\varepsilon}(A|R)_{\rho} := \max_{\sigma_R \in \mathcal{S}(R)} -D_{\max}^{\varepsilon}(\rho_{AR} || \mathbb{1}_A \otimes \sigma_R)$ is the conditional min-entropy and $\tilde{\rho}_{AR} := \rho[\text{id}, \Delta]$ is the dephased CQ state in the protocol.

Remark 1 On the one hand, the randomness extraction protocol without using incoherent operations (e.g. the one considered in [ZLY⁺19, Figure 1.(b)]) is too restrictive, as such a framework does not make good use of free resources at hand. On the other hand, an extraction protocol that does not consider Eve's attack on the free operation Λ is too trivial because Alice can generate an arbitrary amount of randomness by using a free replacer channel $\Lambda(\cdot) = \pi_C$. Hence, the setup in Figure 1 contributes to a reasonable randomness extraction framework in the context of quantum coherence theory.

Remark 2 The randomness extraction framework proposed here is also closely related to the one in [HZ18]. But we should note the following subtle differences: (i) the independence of the resulting randomness is quantified using the trace distance in [HZ18] instead of the purified distance we use in (12). Though the trace distance can give us the nice property of universal composability (see e.g. [Ren05]), the choice of purified distance in (12) is crucial for obtaining the exact relation between coherence distillation and randomness extraction in the next section, which then becomes a key ingredient to proving the second order expansions; (ii) while the paper [HZ18] has a discussion on the large block length regime, its analysis is focused on the exponential decreasing rate for the amount of the leaked information ε , but we will put more focus on the rate of extractable randomness in the one-shot and large block length regime with constant ε .

Also, the references [YZCM15] and [HZ18, Section VI] address the randomness extraction via incoherent operations but with additional constraints on Eve, which are different from our setting here.

Remark 3 Note that performing incoherent unitary operations in the first step does not make any difference with the protocol by identity map id . This justifies our consideration of general incoherent operations. More precisely, for any incoherent unitary U , it holds $\ell_{\text{id}}^{\varepsilon}(\rho_A) = \ell_{\mathcal{U}}^{\varepsilon}(\rho_A)$ with $\mathcal{U}(\cdot) := U(\cdot)U^{\dagger}$. To see this, recall that any incoherent unitary on \mathcal{H}_A can be written as $U_A = \sum_{a \in \mathcal{A}} e^{i\theta_a} |g(a)\rangle\langle a|$ with a permutation g and phase factors $e^{i\theta_a}$ [SAP17, Section II.A.2]. Then a direct calculation gives that $\rho[\mathcal{U}, \Delta, f] = \sum_{a \in \mathcal{A}} p_a |f(g(a))\rangle\langle f(g(a))| \otimes |\psi_a\rangle\langle \psi_a| = \rho[\text{id}, \Delta, f \circ g]$, implying the equivalence of extraction protocols (\mathcal{U}, Δ, f) and $(\text{id}, \Delta, f \circ g)$. Thus $\ell_{\text{id}}^{\varepsilon}(\rho_A) = \ell_{\mathcal{U}}^{\varepsilon}(\rho_A)$ follows by definition.

4 Relation between coherence distillation and randomness extraction

In this section we present an exact relation between coherence distillation and incoherent randomness extraction, the proof of which showcases a nice one-to-one correspondence between coherence distillation protocols and incoherent randomness extraction protocols.

We first present a technical result that will be used in the proof of the one-shot relation.

Proposition 1 (Main technical result) *For any quantum state $\rho_A \in \mathcal{S}(A)$, error tolerance $\varepsilon \in [0, 1]$ and incoherent randomness extraction protocol (id, Δ, f) such that $d_{\text{sec}}(\rho[\text{id}, \Delta, f]_{LR}|R) \leq \varepsilon$, there exists a quantum operation $\Gamma : \mathcal{S}(A) \rightarrow \mathcal{S}(L)$ such that $\Gamma \in \text{DIIO}$ and $P(\Gamma_{A \rightarrow L}(\rho_A), \Psi_L) \leq \varepsilon$.*

Proof A detailed proof is given at the end of this section. ■

Theorem 2 (One-shot relation) For any quantum state $\rho_A \in \mathcal{S}(A)$, error tolerance $\varepsilon \in [0, 1]$ and free operation class $\mathcal{O} \in \{\text{MIO}, \text{DIO}, \text{IO}, \text{DIIIO}\}$, the following equation holds

$$C_{d,\mathcal{O}}^\varepsilon(\rho_A) = \ell_{\mathcal{O}}^\varepsilon(\rho_A). \quad (16)$$

Remark 4 Recall that there is an exact characterization of one-shot distillable coherence under MIO and DIO operations [RFA18, Proposition 2]. Together with the above (16), we have the following relations

$$C_{d,\text{MIO}}^\varepsilon(\rho_A) = C_{d,\text{DIO}}^\varepsilon(\rho_A) = \ell_{\text{MIO}}^\varepsilon(\rho_A) = \ell_{\text{DIO}}^\varepsilon(\rho_A) = \min_{\substack{X=\Delta(X) \\ \text{Tr } X=1}} D_H^\varepsilon(\rho_A \| X) - \delta, \quad (17)$$

where the minimum is taken over all Hermitian operators X on \mathcal{H}_A satisfying the conditions and $\delta \geq 0$ is the least number such that the solution corresponds to the logarithm of an integer.

Remark 5 Note that a one-shot relation between distillable coherence and extractable randomness has appeared in [ZLY⁺19, Equation (80)]. Unlike the precise equation in (16), the relation in [ZLY⁺19] is given in the form of one-shot lower and upper bounds with unmatched error dependence and additional correction terms. However, the clean form in (16) plays a pivotal role in deriving the second order expansions where the error dependence matters.

[Proof of Theorem 2] We first show the direction $\ell_{\mathcal{O}}^\varepsilon(\rho_A) \geq C_{d,\mathcal{O}}^\varepsilon(\rho_A)$. Denote $C_{d,\mathcal{O}}^\varepsilon(\rho_A) = \log |C|$ and suppose that this rate is achieved by a free operation $\Lambda : \mathcal{S}(A) \rightarrow \mathcal{S}(C)$ such that $P(\Lambda(\rho_A), \Psi_C) \leq \varepsilon$. Consider a randomness extraction protocol $(\Lambda, \Delta, \text{id})$. Note that $\rho[\Lambda]_{CER}$ is a purification of $\Lambda(\rho_A)$. By Uhlmann's theorem [Uhl76] there exists an extension of Ψ_C , denoted as $\Psi_C \otimes \sigma_{ER}^*$, such that $P(\Lambda(\rho_A), \Psi_C) = P(\rho[\Lambda]_{CER}, \Psi_C \otimes \sigma_{ER}^*)$. Then we have

$$d_{\text{sec}}(\rho[\Lambda, \Delta, \text{id}]_{CER} | ER) = \min_{\sigma_{ER} \in \mathcal{S}(ER)} P(\Delta_C(\rho[\Lambda]_{CER}), \pi_C \otimes \sigma_{ER}) \quad (18)$$

$$\leq P(\Delta_C(\rho[\Lambda]_{CER}), \pi_C \otimes \sigma_{ER}^*) \quad (19)$$

$$= P(\Delta_C(\rho[\Lambda]_{CER}), \Delta_C(\Psi_C \otimes \sigma_{ER}^*)) \quad (20)$$

$$\leq P(\rho[\Lambda]_{CER}, \Psi_C \otimes \sigma_{ER}^*) \quad (21)$$

$$= P(\Lambda(\rho_A), \Psi_C) \quad (22)$$

$$\leq \varepsilon, \quad (23)$$

where the second equality follows by $\Delta_C(\Psi_C) = \pi_C$, the second inequality follows by the data-processing inequality of purified distance, the third equality follows from the assumption of σ_{ER}^* . Thus we know that $\log |C|$ is an achievable randomness extraction rate, which implies $\ell_{\mathcal{O}}^\varepsilon(\rho_A) \geq \log |C| = C_{d,\mathcal{O}}^\varepsilon(\rho_A)$.

For the other direction, we denote $\ell_{\mathcal{O}}^\varepsilon(\rho_A) = \log |L|$ and suppose that this rate is achieved by an extraction protocol (Λ, Δ, f) with $\Lambda \in \mathcal{O}$. Notice that applying the protocol (Λ, Δ, f) on quantum state ρ_A is the same as applying a protocol (id, Δ, f) on $\Lambda(\rho_A)$ with purification $\rho[\Lambda]_{CER}$ and reference system ER . By Proposition 1 there exists a quantum operation $\Gamma : \mathcal{S}(C) \rightarrow \mathcal{S}(L)$ such that $\Gamma \in \text{DIIIO}$ and $P(\Gamma(\Lambda(\rho_A)), \Psi_L) \leq \varepsilon$. Since $\Gamma \in \text{DIIIO} \subseteq \mathcal{O}$ and $\Lambda \in \mathcal{O}$, we have $\Gamma \circ \Lambda \in \mathcal{O}$ and this operation achieves the distillation rate $\log |L|$. This implies $C_{d,\mathcal{O}}^\varepsilon(\rho_A) \geq \log |L| = \ell_{\mathcal{O}}^\varepsilon(\rho_A)$ and completes the proof. ■

[Proof of Proposition 1] Let σ_R^* be a quantum state that attains the minimum in

$$d_{\text{sec}}(\rho[\text{id}, \Delta, f]_{LR} | R) = \min_{\sigma_R \in \mathcal{S}(R)} P(\rho[\text{id}, \Delta, f]_{LR}, \pi_L \otimes \sigma_R). \quad (24)$$

Let $|\phi^*\rangle_{AR}$ on $\mathcal{H}_A \otimes \mathcal{H}_R$ be a purification of σ_R^* . Thus we have

$$F(\rho[\text{id}, \Delta, f]_{LR}, \pi_L \otimes \sigma_R^*) \geq \sqrt{1 - \varepsilon^2}. \quad (25)$$

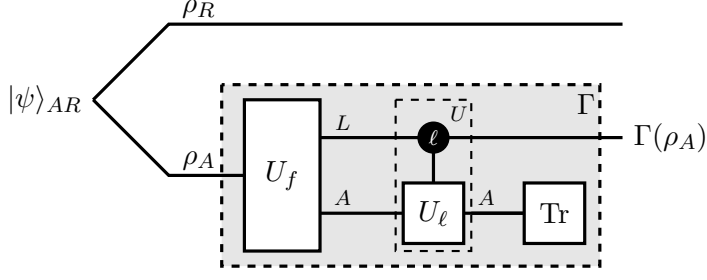


Figure 2: Schematic diagram of the DIIO distillation protocol in Proposition 1. $|\psi\rangle_{AR}$ is a purification of ρ_A . The isometry U_f is incoherent. The unitary U , represented by the inner dashed box, is a controlled unitary but it is not necessarily incoherent. The overall operation Γ , represented by the outer dashed box, belongs to DIIO.

Define the incoherent isometry U_f from \mathcal{H}_A to $\mathcal{H}_L \otimes \mathcal{H}_A$ as

$$U_f|a\rangle_A := |f(a)\rangle_L \otimes |a\rangle_A. \quad (26)$$

We choose normalized vectors $|\phi_\ell\rangle_{AR}$ and normalization factors r_ℓ such that

$$U_f|\psi\rangle_{AR} = \sum_{\ell \in \mathcal{L}} \sqrt{r_\ell} |\ell\rangle_L \otimes |\phi_\ell\rangle_{AR} \quad \text{and} \quad \sqrt{r_\ell} |\phi_\ell\rangle_{AR} := \sum_{a \in \mathcal{A}: f(a)=\ell} \sqrt{p_a} |a\rangle_A \otimes |\psi_a\rangle_R. \quad (27)$$

By Uhlmann's theorem [Uhl76] there exists a unitary U_ℓ on \mathcal{H}_A such that

$$F(\text{Tr}_A |\phi_\ell\rangle\langle\phi_\ell|_{AR}, \sigma_R^*) = F(U_\ell |\phi_\ell\rangle_{AR}, |\phi^*\rangle_{AR}) = \langle\phi^*|U_\ell|\phi_\ell\rangle. \quad (28)$$

Take $U := \sum_{\ell \in \mathcal{L}} |\ell\rangle\langle\ell|_L \otimes U_\ell$. We have

$$F(UU_f|\psi\rangle_{AR}, |\Psi_L\rangle \otimes |\phi^*\rangle_{AR}) \stackrel{(a)}{=} F\left(\sum_{\ell \in \mathcal{L}} \sqrt{r_\ell} |\ell\rangle \otimes U_\ell |\phi_\ell\rangle, \sum_{\ell \in \mathcal{L}} \frac{1}{\sqrt{|\mathcal{L}|}} |\ell\rangle \otimes |\phi^*\rangle\right) \quad (29)$$

$$\stackrel{(b)}{=} \sum_{\ell \in \mathcal{L}} \sqrt{r_\ell} \frac{1}{\sqrt{|\mathcal{L}|}} F(U_\ell |\phi_\ell\rangle, |\phi^*\rangle) \quad (30)$$

$$\stackrel{(c)}{=} \sum_{\ell \in \mathcal{L}} \sqrt{r_\ell} \frac{1}{\sqrt{|\mathcal{L}|}} F(\text{Tr}_A |\phi_\ell\rangle\langle\phi_\ell|, \sigma_R^*) \quad (31)$$

$$\stackrel{(d)}{=} F\left(\sum_{\ell \in \mathcal{L}} r_\ell |\ell\rangle\langle\ell| \otimes \text{Tr}_A |\phi_\ell\rangle\langle\phi_\ell|, \sum_{\ell \in \mathcal{L}} \frac{1}{|\mathcal{L}|} |\ell\rangle\langle\ell| \otimes \sigma_R^*\right) \quad (32)$$

$$\stackrel{(e)}{=} F(\rho[\text{id}, \Delta, f]_{LR}, \pi_L \otimes \sigma_R^*), \quad (33)$$

where (a) follows by definition, (b) and (d) follow from Lemma 4, (c) follows from Eq. (28) and (e) follows from the fact that $\rho[\text{id}, \Delta, f]_{LR} = \sum_{\ell \in \mathcal{L}} r_\ell |\ell\rangle\langle\ell| \otimes \text{Tr}_A |\phi_\ell\rangle\langle\phi_\ell|$. We construct a quantum operation $\Gamma_{A \rightarrow L}(\cdot) := \text{Tr}_A [UU_f(\cdot)U_f^\dagger U^\dagger]$. Then $\Gamma_{A \rightarrow L}(\rho_A) = \text{Tr}_{AR} [UU_f|\psi\rangle\langle\psi|_{AR}U_f^\dagger U^\dagger]$ and we can check that

$$F(\Gamma_{A \rightarrow L}(\rho_A), \Psi_L) \geq F(UU_f|\psi\rangle_{AR}, |\Psi_L\rangle \otimes |\phi^*\rangle_{AR}) = F(\rho[\text{id}, \Delta, f]_{LR}, \pi_L \otimes \sigma_R^*) \geq \sqrt{1 - \varepsilon^2}, \quad (34)$$

where the first inequality follows by the data-processing inequality of quantum fidelity under Tr_{AR} , the equality follows by (33) and the second inequality follows from (25). This implies that $P(\Gamma_{A \rightarrow L}(\rho_A), \Psi_L) \leq \varepsilon$.

It remains to check $\Gamma \in \text{DIIO}$. Note that Γ admits a Kraus decomposition $\Gamma(\cdot) = \sum_{a \in \mathcal{A}} K_a U_f(\cdot) U_f^\dagger K_a^\dagger$ with operators $K_a = \langle a|U$. For any computational basis $|x\rangle$ and any $a \in \mathcal{A}$, we have

$$K_a U_f |x\rangle \langle x| U_f^\dagger K_a^\dagger = |\langle a|U_{f(x)}|x\rangle|^2 |f(x)\rangle \langle f(x)| \in \mathcal{I}^{**}, \quad (35)$$

by direct calculation. Thus $\Gamma \in \text{IO}$. For any computational basis $|x\rangle$ and $|y\rangle$, we can first check that

$$\Gamma(|x\rangle\langle y|) = \langle y|U_{f(x)}^\dagger U_{f(y)}|x\rangle |f(x)\rangle \langle f(y)|. \quad (36)$$

Thus it holds

$$\Gamma(\Delta(|x\rangle\langle y|)) = \Gamma(\delta_{x,y}|x\rangle\langle x|) = \delta_{x,y}\Gamma(|x\rangle\langle x|) = \delta_{x,y}|f(x)\rangle\langle f(x)|, \quad (37)$$

and

$$\Delta(\Gamma(|x\rangle\langle y|)) = \delta_{f(x),f(y)}\langle y|U_{f(x)}^\dagger U_{f(y)}|x\rangle|f(x)\rangle\langle f(x)| = \delta_{f(x),f(y)}\delta_{x,y}|f(x)\rangle\langle f(x)| = \delta_{x,y}|f(x)\rangle\langle f(x)|. \quad (38)$$

Combining (37) and (38), we have $\Gamma \circ \Delta = \Delta \circ \Gamma$, indicating that $\Gamma \in \text{DIO}$. Finally we have $\Gamma \in \text{DIIO}$. \blacksquare

5 Second order analysis

In this section we discuss the second order expansions of distillable coherence and extractable randomness. The technical statement is presented as follows.

Theorem 3 (Second order expansion) *For any quantum state $\rho_A \in \mathcal{S}(A)$, error tolerance $\varepsilon \in (0, 1)$ and free operation class $\mathcal{O} \in \{\text{MIO}, \text{DIO}, \text{IO}, \text{DIIO}\}$, the following second order expansion holds*

$$C_{d,\mathcal{O}}^\varepsilon(\rho_A^{\otimes n}) = \ell_{\mathcal{O}}^\varepsilon(\rho_A^{\otimes n}) = nD(\rho_A \|\Delta(\rho_A)) + \sqrt{nV(\rho_A \|\Delta(\rho_A))} \Phi^{-1}(\varepsilon^2) + O(\log n), \quad (39)$$

where Φ^{-1} denotes the inverse of the cumulative distribution function of a standard normal random variable.

Remark 6 Comparing the second order expansion of $\ell_{\text{id}}^\varepsilon(\rho_A^{\otimes n})$ in [TH13, Corollary 16] and the result above, we can conclude that a general incoherent randomness extraction protocol (Λ, Δ, f) has no advantage over the protocol (id, Δ, f) in the sense that they lead to the same first order asymptotics [HZ18] and the second order asymptotics of extractable randomness.

Remark 7 The distillable coherence under MIO/DIO/IO/DIIO not only have the same first order asymptotics as observed in [WY16, RFWA18, ZLY⁺19] but also have the same second order asymptotics, indicating that they are equivalently powerful for coherence distillation in the large block length regime.

Remark 8 For any quantum state $\rho = \sum_{i,j} \rho_{ij} |i\rangle\langle j|_A$ written in the computational basis, we can assign it to a bipartite maximally correlated state $\rho_{\text{mc}} := \sum_{i,j} \rho_{ij} |i\rangle\langle j|_A \otimes |i\rangle\langle j|_B$. The second order expansion of distillable entanglement of $\rho_{\text{mc}}^{\otimes n}$ under local operations and classical communication (LOCC) is also given by $nD(\rho \|\Delta(\rho)) + \sqrt{nV(\rho \|\Delta(\rho))} \Phi^{-1}(\varepsilon^2) + O(\log n)$ [FWTD19, Proposition 10]. Together with the result in Theorem 3, the coincidence of these second order expansions leads to a new evidence to the long-standing conjecture (see e.g. [SAP17, Section II.D]) that any incoherent operation acting on a state ρ is equivalent to a LOCC operation acting on the associated maximally correlated state ρ_{mc} .

[Proof of Theorem 3] The converse part is a direct consequence of a known result in [RFWA18, Proposition 2]. That is, for any quantum state ρ_A it holds $C_{d,\text{MIO}}^\varepsilon(\rho_A) = \min_{X=\Delta(X), \text{Tr } X=1} D_H^{\varepsilon^2}(\rho_A \| X) - \delta$, where δ is the least number such that the solution on the r.h.s. corresponds to the logarithm of an integer. Choosing a feasible solution X as $\Delta(\rho)$, we have $C_{d,\text{MIO}}^\varepsilon(\rho_A) \leq D_H^{\varepsilon^2}(\rho_A \|\Delta(\rho_A))$. Given the i.i.d. state $\rho_A^{\otimes n}$, we have

$$C_{d,\mathcal{O}}^\varepsilon(\rho_A^{\otimes n}) \leq C_{d,\text{MIO}}^\varepsilon(\rho_A^{\otimes n}) \leq D_H^{\varepsilon^2}(\rho_A^{\otimes n} \|\Delta(\rho_A)^{\otimes n}), \quad (40)$$

where the first inequality follows by the fact that $\mathcal{O} \subseteq \text{MIO}$. Expanding the r.h.s. of (40) via formula (5) we have the second order upper bound.

By the one-shot relation in Theorem 2 and (15), we have for any $\eta \in (0, \varepsilon]$,

$$C_{d,\mathcal{O}}^\varepsilon(\rho_A) = \ell_{\mathcal{O}}^\varepsilon(\rho_A) \geq \ell_{\text{id}}^\varepsilon(\rho_A) \geq H_{\min}^{\varepsilon-\eta}(A|R)_{\tilde{\rho}} + 4 \log \eta - 3, \quad (41)$$

where $\tilde{\rho}_{AR} := \rho[\text{id}, \Delta]$ and the first inequality follows by definition. By definition we also have $H_{\min}^{\varepsilon-\eta}(A|R)_{\tilde{\rho}} \geq -D_{\max}^{\varepsilon-\eta}(\tilde{\rho}_{AR} \|\mathbb{1}_A \otimes \tilde{\rho}_R)$. Hence, given the i.i.d. state $\rho_A^{\otimes n}$, we have

$$C_{d,\mathcal{O}}^\varepsilon(\rho_A^{\otimes n}) \geq -D_{\max}^{\varepsilon-\eta}(\tilde{\rho}_{AR}^{\otimes n} \|\mathbb{1}_A \otimes \tilde{\rho}_R^{\otimes n}) + 4 \log \eta - 3. \quad (42)$$

Choosing $\eta = 1/\sqrt{n}$ and expanding the r.h.s. of (42) via formula (6), we have

$$C_{d,\mathcal{O}}^\varepsilon(\rho_A^{\otimes n}) \geq -nD(\tilde{\rho}_{AR}\|\mathbb{1}_A \otimes \tilde{\rho}_R) + \sqrt{nV(\tilde{\rho}_{AR}\|\mathbb{1}_A \otimes \tilde{\rho}_R)} \Phi^{-1}(\varepsilon^2) + O(\log n). \quad (43)$$

Here we used a fact that $\Phi^{-1}((\varepsilon - 1/\sqrt{n})^2) = \Phi^{-1}(\varepsilon^2) + O(1/\sqrt{n})$ in the expansion, which follows as Φ^{-1} is continuously differentiable around $\varepsilon > 0$. Finally the proof is completed by noticing that $-D(\tilde{\rho}_{AR}\|\mathbb{1}_A \otimes \tilde{\rho}_R) = D(\rho_A\|\Delta(\rho_A))$ and $V(\tilde{\rho}_{AR}\|\mathbb{1}_A \otimes \tilde{\rho}_R) = V(\rho_A\|\Delta(\rho_A))$. These two equations are calculated in Lemma 5. ■

6 Strong converse property

The direct part of quantum coherence distillation states that for any rate below the optimal rate, there is a corresponding distillation protocol that accomplishes the task successfully. More precisely, if we denote the transformation error in the protocol for n uses of the underlying resource by ε , then for any rate below the optimal rate there exists a protocol, whose transformation error ε vanishes in the asymptotic limit $n \rightarrow +\infty$. Such rates are called achievable, and the optimal rate is defined as the supremum over all achievable rates. In contrast, the converse part states that for any distillation protocol with a rate above the optimal rate, the error does not vanish asymptotically, that is, it is bounded away from 0 in the asymptotic limit $n \rightarrow +\infty$. This is usually called weak converse. In principle, it leaves open the possibility of a trade-off between error and rate of a protocol. However, the strong converse property rules out such a possibility, stating that for any distillation protocol with a rate above the optimal rate, the corresponding transformation error ε incurred in the protocol converges to one. In other words, such protocols become worse with increasing block length n , and eventually fail with certainty in the asymptotic limit.

The strong converse property of coherence distillation has been pointed out by [ZLY⁺19, Theorem 16]. In this part we give an alternative proof and showcase a standard argument how a second order result automatically implies the strong converse property. For simplicity, we denote $C_r(\rho) := D(\rho\|\Delta(\rho))$ and $V_r(\rho) := V(\rho\|\Delta(\rho))$. For any achievable rate R_n , we have $R_n \leq \frac{1}{n}C_{d,\mathcal{O}}^\varepsilon(\rho^{\otimes n})$. By Theorem 3, we have

$$R_n \leq C_r(\rho) + \sqrt{\frac{V_r(\rho)}{n}} \Phi^{-1}(\varepsilon^2) + f(n) \quad \text{with} \quad f(n) \in O\left(\frac{\log n}{n}\right). \quad (44)$$

Rearranging (44) and using monotonicity of Φ yields

$$\varepsilon^2 \geq \Phi\left(\sqrt{\frac{n}{V_r(\rho)}}(R_n - C_r(\rho)) + g(n)\right) \quad \text{with} \quad g(n) = -\frac{\sqrt{n}f(n)}{\sqrt{V_r(\rho)}}. \quad (45)$$

Thus $\lim_{n \rightarrow +\infty} g(n) = 0$. Note that $\lim_{x \rightarrow +\infty} \Phi(x) = 1$. For any achievable rate $R_n > C_r(\rho)$, the argument in (45) diverges to $+\infty$ and thus we have $\varepsilon \rightarrow 1$ as $n \rightarrow \infty$. This implies the strong converse property of coherence distillation under $\mathcal{O} \in \{\text{MIO}, \text{DIO}, \text{IO}, \text{DIIO}\}$. Similar argument works for the incoherent randomness extraction.

7 Conclusions

We have introduced a variant of randomness extraction framework in the context of quantum coherence theory, establishing an exact relation between this cryptographic task and the operational task of quantum coherence distillation. Based on this relation, we have given a finite block length analysis on these tasks, providing in particular explicit second order expansions of distillable coherence and extractable randomness under a diverse range of free operations. It turns out that the operation classes MIO/DIO/IO/DIIO all admit the same second order expansions, indicating their equivalent power for coherence distillation and randomness extraction not only in the asymptotic regime but also in the large block length regime. Our work has initiated the first second order analysis in coherence theory, filling an important gap in the literature. As coherence is generically undistillable under SIO/PIO [LRA19, Lam19], our results have completed the second order analysis on distillable coherence under all major classes of free operations.

As a by-product of second order expansions, we have also shown an alternative proof of the strong converse property of coherence distillation and randomness extraction. But we left open the analysis on their strong

converse exponents (the exact rate of error measure converges to one when the achievable rate is over the optimal rate) and the error exponents (the exact rate of error measure decays to zero when the achievable rate is below the optimal rate), the latter of which has been studied in [HZ18, Section VII] for randomness extraction with trace norm being the security measure. It would be interesting to see if the precise connection between randomness extraction and coherence distillation can provide insights into these problems.

The coincidence of the second order expansions of $\ell_{\mathcal{O}}^{\varepsilon}(\rho_A^{\otimes n})$ and $\ell_{\text{id}}^{\varepsilon}(\rho_A^{\otimes n})$ indicates that optimizing the free incoherent operations before the incoherent measurement can improve the extractable randomness by the order $O(\log n)$ at most. It would be interesting to further investigate if there is any advantage of performing incoherent operations in the third or higher order terms.

Finally, as a reverse problem of coherence distillation, the coherence cost considers the minimum number of coherent bits required to prepare a quantum state. It is known, for example, that the first order asymptotics of coherence cost under IO operations is given by the coherence information [WY16]. But what is the second order expansion? Recall the important role of randomness extraction framework in our second order analysis. For coherence cost, we may consider a randomness extraction scenario with Eve having limited power. Such a scenario has been studied in [YZCM15] and [HZ18, Section VI] and the corresponding randomness extraction rate happens to coincide with the coherence information.

Acknowledgements. We thank Anurag Anshu and Xin Wang for discussions about the second order expansion of distillable coherence. KF thanks the Center for Quantum Computing at Peng Cheng Laboratory for their hospitality while part of this work was done during his visit. MH was supported in part by a JSPS Grant-in-Aid for Scientific Research (A) No.17H01280, (B) No. 16KT0017, the Okawa Research Grant and Kayamori Foundation of Informational Science Advancement.

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Appendix A Technical lemmas

Lemma 4 Let $|u_i\rangle$ and $|v_i\rangle$ be the purification of ρ_i and σ_i such that $F(\rho_i, \sigma_i) = \langle u_i|v_i\rangle$. Then we have

$$F\left(\sum_i p_i|i\rangle\langle i| \otimes \rho_i, \sum_i q_i|i\rangle\langle i| \otimes \sigma_i\right) = \sum_i \sqrt{p_i q_i} F(\rho_i, \sigma_i) = F\left(\sum_i \sqrt{p_i}|i\rangle|u_i\rangle, \sum_i \sqrt{q_i}|i\rangle|v_i\rangle\right). \quad (46)$$

Proof These two equalities can be check by definition. \blacksquare

Suppose $\rho = \sum_x r_x|v_x\rangle\langle v_x|$ and $\sigma = \sum_y s_y|u_y\rangle\langle u_y|$ are the eigenvalue decompositions. Their Nussbaum and Szkoła's distributions [NS09] are defined as $P_{\rho,\sigma}(x, y) := r_x|\langle v_x|u_y\rangle|^2$ and $Q_{\rho,\sigma}(x, y) := s_y|\langle v_x|u_y\rangle|^2$. These distributions have a very convenient property that the first two moments of $\log P_{\rho,\sigma} - \log Q_{\rho,\sigma}$ under $P_{\rho,\sigma}$ agree with the respective moments of $\log \rho - \log \sigma$ under ρ . Namely, it is easy to verify that

$$D(P_{\rho,\sigma}\|Q_{\rho,\sigma}) = D(\rho\|\sigma) \quad \text{and} \quad V(P_{\rho,\sigma}\|Q_{\rho,\sigma}) = V(\rho\|\sigma). \quad (47)$$

For any two hermitian operators X and Y , we use $\{X \geq Y\}$ to denote the projector onto the space spanned by the eigenvectors of $X - Y$ that corresponds to non-negative eigenvalues. Then the information spectrum is defined as [TH13]

$$D_s^\varepsilon(\rho\|\sigma) := \sup \{x : \text{Tr} \rho\{x \leq 2^x\sigma\} \leq \varepsilon\}. \quad (48)$$

Lemma 5 For any quantum state $\rho_A \in \mathcal{S}(A)$, let $|\psi\rangle_{AR} := \sum_{a \in A} \sqrt{p_a}|a\rangle_A|\psi_a\rangle_R$ be its purification, i.e., $\text{Tr}_R |\psi\rangle\langle\psi|_{AR} = \rho_A$. Denote the dephased CQ state as $\tilde{\rho}_{AR} := \sum_{a \in A} p_a|a\rangle\langle a|_A \otimes |\psi_a\rangle\langle\psi_a|_R$. It holds

$$D_s^\varepsilon(P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}\|Q_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}) = -D_s^{1-\varepsilon}(P_{\rho, \Delta(\rho)}\|Q_{\rho, \Delta(\rho)}), \quad \forall \varepsilon \in [0, 1], \quad (49)$$

$$D(\tilde{\rho}_{AR}\|\mathbb{1}_A \otimes \tilde{\rho}_R) = -D(\rho_A\|\Delta(\rho_A)), \quad (50)$$

$$V(\tilde{\rho}_{AR}\|\mathbb{1}_A \otimes \tilde{\rho}_R) = V(\rho_A\|\Delta(\rho_A)). \quad (51)$$

Proof Let $\tilde{\rho}_R = \sum_r q_r|\varphi_r\rangle\langle\varphi_r|$ be the eigenvalue decomposition. Then the eigenvalue decomposition of $\mathbb{1}_A \otimes \tilde{\rho}_R$ is given by $\mathbb{1}_A \otimes \tilde{\rho}_R = \sum_{a,r} q_r|a\rangle\langle a|_A|\varphi_r\rangle\langle\varphi_r|_R$. By definition we have $P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a', (a, r)) = p_{a'}|\langle\psi_{a'}|\varphi_r\rangle|^2 \delta_{a',a}$ and $Q_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a', (a, r)) = q_r|\langle\psi_{a'}|\varphi_r\rangle|^2 \delta_{a',a}$. Hence it is sufficient to discuss the case when $a' = a$. That is, we have

$$P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a, r) = p_a|\langle\psi_a|\varphi_r\rangle|^2 \quad \text{and} \quad Q_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a, r) = q_r|\langle\psi_a|\varphi_r\rangle|^2. \quad (52)$$

Note that $\tilde{\rho}_R = \text{Tr}_A \tilde{\rho}_{AR} = \text{Tr}_A |\psi\rangle\langle\psi|_{AR}$. This implies that $\tilde{\rho}_R$ and ρ_A are two marginal states of $|\psi\rangle_{AR}$, and thus have the same eigenvalues. Therefore, we can denote the eigenvalue decomposition of ρ_A as $\rho_A = \sum_r q_r|u_r\rangle\langle u_r|$. Since $\rho_A = \text{Tr}_R |\psi\rangle\langle\psi|_{AR} = \sum_{a,a'} \sqrt{p_a p_{a'}}|\psi_{a'}\langle\psi_a|_A|a\rangle\langle a'|$, we have $\Delta(\rho_A) = \sum_a p_a|a\rangle\langle a|_A$, which is an eigenvalue decomposition itself. Then by definition we have

$$P_{\rho, \Delta(\rho)}(r, a) = q_r|\langle u_r|a\rangle|^2 \quad \text{and} \quad Q_{\rho, \Delta(\rho)}(r, a) = p_a|\langle u_r|a\rangle|^2. \quad (53)$$

Moreover, we have the relation that $|\psi\rangle_{AR} = \sum_r \sqrt{q_r}|u_r\rangle_A|\varphi_r\rangle_R$ as they are two pure state with the same marginals. Then it holds $\langle\varphi_r|\psi\rangle\langle\psi|\varphi_r\rangle = q_r|u_r\rangle\langle u_r|$. We further have

$$P_{\rho, \Delta(\rho)}(r, a) = q_r|\langle u_r|a\rangle|^2 = q_r\langle a|u_r\rangle\langle u_r|a\rangle = |\langle a|\langle\varphi_r|\psi\rangle|^2 = p_a|\langle\varphi_r|\psi_a\rangle|^2 = P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a, r). \quad (54)$$

Hence we can check that

$$D_s^\varepsilon(P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}\|Q_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}) = \sup \{x : P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}\{(a, r) | \log P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a, r) - \log Q_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a, r) \leq x\} \leq \varepsilon\} \quad (55)$$

$$= \sup \{x : P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}\{(a, r) | \log p_a - \log q_r \leq x\} \leq \varepsilon\} \quad (56)$$

$$= \sup \{x : P_{\rho, \Delta(\rho)}\{(r, a) | \log p_a - \log q_r \leq x\} \leq \varepsilon\} \quad (57)$$

$$= \sup \{x : P_{\rho, \Delta(\rho)}\{(r, a) | \log Q_{\rho, \Delta(\rho)}(r, a) - \log P_{\rho, \Delta(\rho)}(r, a) \leq x\} \leq \varepsilon\} \quad (58)$$

$$= -D_s^{1-\varepsilon}(P_{\rho, \Delta(\rho)}\|Q_{\rho, \Delta(\rho)}), \quad (59)$$

where the first and last equalities follows by definition, the second equality follows by (52), the third equality follows by (54), the fourth equality follows by (53). This concludes the proof of (49). Similarly, we have

$$D(\tilde{\rho}_{AR} \|\mathbb{1}_A \otimes \tilde{\rho}_R) = D(P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R} \| Q_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}) \quad (60)$$

$$= \sum_{a,r} P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a, r) [\log P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a, r) - \log Q_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}(a, r)] \quad (61)$$

$$= \sum_{a,r} P_{\rho, \Delta(\rho)}(r, a) [\log p_a - \log q_r] \quad (62)$$

$$= \sum_{a,r} P_{\rho, \Delta(\rho)}(r, a) [\log Q_{\rho, \Delta(\rho)}(r, a) - \log P_{\rho, \Delta(\rho)}(r, a)] \quad (63)$$

$$= -D(P_{\rho, \Delta(\rho)} \| Q_{\rho, \Delta(\rho)}) \quad (64)$$

$$= -D(\rho_A \| \Delta(\rho_A)), \quad (65)$$

where the first and last equalities follow by the relation (47), the second and fifth equalities follow by definition, the third and fourth inequalities follow by (52), (53), (54). This completes the proof of (50). We can also check that $V(\tilde{\rho}_{AR} \|\mathbb{1}_A \otimes \tilde{\rho}_R) = V(\rho_A \| \Delta(\rho_A))$ in a similar way. \blacksquare

We further employ the following definition. For a given positive semidefinite operator σ , we denote the number of distinct eigenvalues of σ by $\nu(\sigma)$. Define the number $\lambda(\sigma) := \log \lambda_{\max}(\sigma) - \log \lambda_{\min}(\sigma)$ where λ_{\max} is the maximum and λ_{\min} the minimum eigenvalue of σ . Denote $\theta(\sigma) := \min\{2\lceil \lambda(\sigma) \rceil, \nu(\sigma)\}$. Recall the relations between one-shot information quantities as follows.

Lemma 6 (Thm. 14 in [TH13]) For any $\rho \in \mathcal{S}$ and $\sigma \in \mathcal{P}$, $0 < \varepsilon < 1$ and $0 < \delta < \min\{\varepsilon, 1 - \varepsilon\}$, it holds

$$D_{\max}^{\sqrt{1-\varepsilon}}(\rho \| \sigma) \leq D_s^{\varepsilon+\delta}(P_{\rho, \sigma} \| Q_{\rho, \sigma}) + c_1(\sigma, \varepsilon, \delta), \quad (66)$$

$$D_H^\varepsilon(\rho \| \sigma) \leq D_s^{\varepsilon+\delta}(P_{\rho, \sigma} \| Q_{\rho, \sigma}) + c_2(\sigma, \varepsilon, \delta), \quad (67)$$

with $c_1(\sigma, \varepsilon, \delta) := \log \theta(\sigma) - \log(\delta(1 - \varepsilon))$, $c_2(\sigma, \varepsilon, \delta) := \log \theta(\sigma) + \log(2^8(\varepsilon + \delta)) - \log(\delta^4(1 - \varepsilon - \delta))$.

Based on the above relations, we can obtain the following result.

Lemma 7 For any quantum state $\rho_A \in \mathcal{S}(A)$, let $|\psi\rangle_{AR} := \sum_{a \in \mathcal{A}} \sqrt{p_a} |a\rangle_A |\psi_a\rangle_R$ be its purification, i.e., $\text{Tr}_R |\psi\rangle\langle\psi|_{AR} = \rho_A$. Denote the dephased CQ state as $\tilde{\rho}_{AR} := \sum_{a \in \mathcal{A}} p_a |a\rangle\langle a|_A \otimes |\psi_a\rangle\langle\psi_a|_R$. Assume $0 < \varepsilon < 1$ and $0 < \delta < \min\{\varepsilon^2/3, 1 - \varepsilon^2\}$. It holds

$$H_{\min}^\varepsilon(A|R)_{\tilde{\rho}} \geq D_H^{\varepsilon^2-2\delta}(\rho_A \| \Delta(\rho_A)) - c(\rho_A, \varepsilon, \delta), \quad (68)$$

with the correction term $c(\rho_A, \varepsilon, \delta) = \log \theta(\rho_A) + \log \theta(\Delta(\rho_A)) + \log(2^8(\varepsilon^2 - \delta)) - \log(\delta^5 \varepsilon^2(1 - \varepsilon^2 + \delta))$.

Proof We have

$$H_{\min}^\varepsilon(A|R)_{\tilde{\rho}} \geq -D_{\max}^\varepsilon(\tilde{\rho}_{AR} \|\mathbb{1}_A \otimes \tilde{\rho}_R) \quad (69)$$

$$\geq -D_s^{1-\varepsilon^2+\delta}(P_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R} \| Q_{\tilde{\rho}_{AR}, \mathbb{1}_A \otimes \tilde{\rho}_R}) - c_1(\rho_A, 1 - \varepsilon^2, \delta) \quad (70)$$

$$= D_s^{\varepsilon^2-\delta}(P_{\rho, \Delta(\rho)} \| Q_{\rho, \Delta(\rho)}) - c_1(\rho_A, 1 - \varepsilon^2, \delta) \quad (71)$$

$$\geq D_H^{\varepsilon^2-2\delta}(\rho \| \Delta(\rho)) - c_2(\Delta(\rho), \varepsilon^2 - 2\delta, \delta) - c_1(\rho_A, 1 - \varepsilon^2, \delta). \quad (72)$$

The first line follows by definition. The second line follows by (66) and the fact that $\theta(\mathbb{1}_A \otimes \tilde{\rho}_R) = \theta(\tilde{\rho}_R)$. Note that $\text{Tr}_A |\psi\rangle\langle\psi|_{AR} = \tilde{\rho}_R$. Thus $\tilde{\rho}_R$ and ρ_A are two marginal states of $|\psi\rangle_{AR}$, implying that they have the same eigenvalues. Then we further have $\theta(\tilde{\rho}_R) = \theta(\rho_A)$. The third line follows by Lemma 5. The last line follows by (67). \blacksquare

Appendix B Distillable coherence and hypothesis testing relative entropy

In this part we show that the one-shot distillable coherence under $\mathcal{O} \in \{\text{MIO}, \text{DIO}, \text{IO}, \text{DIO}\}$ is roughly characterized by $C_{d,\mathcal{O}}^\varepsilon(\rho_A) \approx D_H^{\varepsilon^2}(\rho_A \|\Delta(\rho_A))$ up to some correction terms. This characterization also leads to an alternative proof of the second order expansion in Theorem 3.

Proposition 8 *For any quantum state $\rho_A \in \mathcal{S}(A)$, free operation class $\mathcal{O} \in \{\text{MIO}, \text{DIO}, \text{IO}, \text{DIO}\}$, error tolerance $\varepsilon \in (0, 1)$ and $0 < \eta < \varepsilon$, $0 < \delta < \min\{(\varepsilon - \eta)^2/3, 1 - (\varepsilon - \eta)^2\}$, it holds*

$$D_H^{(\varepsilon-\eta)^2-2\delta}(\rho_A \|\Delta(\rho_A)) - c(\rho_A, \varepsilon, \delta, \eta) \leq C_{d,\mathcal{O}}^\varepsilon(\rho_A) \leq D_H^{\varepsilon^2}(\rho_A \|\Delta(\rho_A)), \quad (73)$$

where $c(\rho_A, \varepsilon, \delta, \eta) = \log \theta(\rho_A) + \log \theta(\Delta(\rho_A)) + \log((\varepsilon - \eta)^2 - \delta) - \log(\delta^5 \eta^4 (\varepsilon - \eta)^2 (1 - (\varepsilon - \eta)^2 + \delta)) + 11$.

Proof The upper bound has been mentioned in (40). According to (41), we have $C_{d,\mathcal{O}}^\varepsilon(\rho_A) \geq H_{\min}^{\varepsilon-\eta}(A|R)_{\tilde{\rho}} + 4 \log \eta - 3$. From Lemma 7 we can further lower bound the conditional min-entropy by the hypothesis testing relative entropy. This gives us the one-shot lower bound stated in (73). ■

Remark 9 Note that $\theta(\sigma^{\otimes n}) \leq 2 \lceil \lambda(\sigma^{\otimes n}) \rceil = 2 \lceil n \lambda(\sigma) \rceil$ which scales at most linearly in n . Choosing η and δ proportional to $1/\sqrt{n}$, we know that the correction term $c(\rho^{\otimes n}, \varepsilon, \delta, \eta) \in O(\log n)$ and $\Phi^{-1}((\varepsilon - \eta)^2 - 2\delta) = \Phi^{-1}(\varepsilon^2) + O(1/\sqrt{n})$. Thus expanding both sides of (73) via formula (5) can also lead to the second order result in Theorem 3. Compared with the one-shot estimation in [ZLY⁺19, Equations (37,46,47)], we can verify that their upper and lower bounds on the one-shot distillable coherence agree in the first order term but disagree in the second order term. In particular, the dependence on ε is qualitatively different in their upper and lower bounds. Thus, one could certainly argue that the bounds in [ZLY⁺19] are not as tight as they should be in the asymptotic limit.